10 Lesson 10

10.1 Warm-up Problems

For each function y, find functions $f(x)$ and $g(x)$ such that $y = f(g(x))$.

- $y = e^{5x}$

$f(x) = e^x$

$g(x) = 5x$

$y = f(g(x)) = e^{5x}$ ✓

- $y = \sqrt[5]{\sin(x) + 4x}$

$f(x) = \sqrt[5]{x}$

$g(x) = \sin(x) + 4x$

$y = f(g(x)) = \sqrt[5]{\sin(x)+4x}$ ✓

10.2 The Chain Rule

Suppose $y = f(g(x))$. Then,

$$y' = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x).$$

Similarly, if we let $u = g(x)$, then

$$y' = \frac{d}{dx}\left[f(g(x))\right] = \frac{df}{du}\frac{du}{dx}.$$

Example 1: Suppose $y = 3(x^2 - x)^4$. Find y'.

$f(x) = 3x^4$

$g(x) = x^2 - x$

$f'(x) = 12x^3$

$g'(x) = 2x - 1$

$y' = f'(g(x))g'(x)$

$= 12(x^2 - x)^3(2x-1)$

$= 12(x(x-1))^3(2x-1)$

$= 12(x-1)^3x^3(2x-1)$

$f(g(x)) = 3(x^2-x)^4$ ✓

Example 2: Suppose $y = \sqrt[3]{6x^2 + 5x - 21}$. Find y'.

$f(x) = \sqrt[3]{x} = x^{1/3}$

$g(x) = 6x^2 + 5x - 21$

$f'(x) = \frac{1}{3}x^{-2/3}$

$g'(x) = 12x + 5$

$y' = f'(g(x))g'(x)$

$= \frac{1}{3}(6x^2+5x-21)^{-2/3}(12x+5)$

$= \dfrac{12x+5}{3(6x^2+5x-21)^{2/3}}$

Example 3: Suppose
$$y = \frac{5}{(5-x^4)^{3/2}}.$$
Find y'.

$y = 5(5-x^4)^{-3/2}$

$f(x) = 5x^{-3/2}$

$g(x) = 5-x^4$

$f'(x) = 5 \cdot -\frac{3}{2} x^{-5/2} = -\frac{15}{2} x^{-5/2}$

$g'(x) = -4x^3$

$y' = f'(g(x))g'(x)$

$= -\frac{15}{2}(5-x^4)^{-5/2} \cdot -4x^3$

$= \dfrac{30x^3}{(5-x^4)^{5/2}}$

Example 4: Suppose
$$y = \left(\frac{3x^2}{2x+4}\right)^3.$$
Find y'.

$f(x) = x^3$

$g(x) = \dfrac{3x^2}{2x+4}$

$f'(x) = 3x^2$

$g'(x) = \dfrac{6x(2x+4) - 2 \cdot 3x^2}{(2x+4)^2}$

$= \dfrac{12x^2 + 24x - 6x^2}{(2x+4)^2}$

$= \dfrac{6x^2+24x}{(2x+4)^2}$

$y' = f'(g(x))g'(x)$

$= 3\left(\dfrac{3x^2}{2x+4}\right)^2\left(\dfrac{6x^2+24x}{(2x+4)^2}\right)$

$= \dfrac{3 \cdot 9x^4 \overbrace{(6x^2+24x)}^{6x(x+4)}}{(2x+4)^4}$

$= \dfrac{\overbrace{162}^{3 \cdot 9 \cdot 6} x^5(x+4)}{(2(x+2))^4}$ $\}= 16(x+2)^4$

$= \dfrac{81x^5(x+4)}{8(x+2)^4}$

Example 5: Suppose $y = 6(3e^x + 2\sin(x) + 5)^3$. Find y'.

$f(x) = 6x^3$ $\quad$ $f'(x) = 18x^2$ $\quad$ $y' = f'(g(x))g'(x)$

$g(x) = 3e^x + 2\sin(x) + 5$ $\quad$ $g'(x) = 3e^x + 2\cos(x)$ $\quad$ $= 18(3e^x + 2\sin(x) + 5)^2(3e^x + 2\cos(x))$