13 Lesson 13

13.1 Implicit Differentiation

Definition: A function is in explicit form if it is written in the form $y = f(x)$, i.e. y is isolated on one side of the equal sign and everything else that does not involve y is on the other side.

Example 1: The function $y = 2x + 7$ is in explicit form. Find $y' = \frac{dy}{dx}$.

$$\frac{dy}{dx} = y' = 2$$

Definition: A function is in implicit form if it is not in explicit form.

Note: Because y is a function of x (i.e. $y = f(x)$), when we take the derivative of y with respect to x, we have that $y' = \frac{dy}{dx}$, not 1.

think $y = f(x)$

Example 1 (revisited): The function $y - 2x = 7$ is in implicit form. Find $y' = \frac{dy}{dx}$.

Taking the derivative of both sides with respect to x,

$$\frac{d}{dx}(y - 2x) = \frac{d}{dx}(7) \Rightarrow \frac{dy}{dx} + \frac{d}{dx}(-2x) = \frac{d}{dx}(7) \Rightarrow \frac{dy}{dx} - 2 = 0 \Rightarrow y' = \frac{dy}{dx} = 2$$

Note: We call the technique used in the previous example *implicit differentiation*. The key is to take the derivative with respect to x of both sides of the equation. This technique is especially useful when we cannot put an equation into explicit form.

think $y = f(x)$

Example 2: Let $3y^2 + 2xy = 4x + 3$. Find $y' = \frac{dy}{dx}$

$$\frac{d}{dx}(3y^2 + 2xy) = \frac{d}{dx}(4x + 3)$$

$$\Rightarrow \underbrace{\frac{d}{dx}(3y^2)}_{6yy'} + \underbrace{\frac{d}{dx}(2xy)}_{2y + y'2x} = \frac{d}{dx}(4x) + \frac{d}{dx}(3)$$

(chain rule)
↳ outer: $3u^2$
↳ inner: y

(product rule)

$$\Rightarrow 6yy' + 2y + y'2x = 4$$

$$\Rightarrow y'(6y + 2x) = 4 - 2y$$

$$\Rightarrow y' = \frac{4 - 2y}{6y + 2x}$$

Example 3: Let

$$3\sin\left(\frac{y}{x}\right) = 7x.$$

Find y'.

Taking the derivative of both sides with respect to x,

$$3\cos\left(\frac{y}{x}\right)\cdot\frac{(y'\cdot x - \overbrace{1y}^{y})}{x^2} = 7 \qquad \text{(chain and quotient rule)}$$

$$\Rightarrow\ 3\cos\left(\frac{y}{x}\right)\frac{y'}{x} - 3\cos\left(\frac{y}{x}\right)\cdot\frac{y}{x^2} = 7$$

$$\Rightarrow\ y' = \frac{7 + 3\cos\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}}{\frac{3}{x}\cos\left(\frac{y}{x}\right)} = \frac{7x^2 + 3\cos\left(\frac{y}{x}\right)y}{3x\cos\left(\frac{y}{x}\right)}$$

mult. top & bottom by x^2

Example 4: Let $e^{6xy} = 7x$. Find y'.

$$e^{6xy}\cdot(6y + y'6x) = 7 \qquad \text{(chain and product rule)}$$

$$\Rightarrow\ e^{6xy}6y + e^{6xy}y'6x = 7$$

$$\Rightarrow\ \frac{7 - 6ye^{6xy}}{6xe^{6xy}} = y'$$

Example 5: Let $8\cos(x)\sin(y) = 5$. Find y'.

$$-8\sin(x)\sin(y) + \cos(y)y'8\cos(x) = 0$$

$$\Rightarrow\ \cos(y)y'8\cos(x) = 8\sin(x)\sin(y)$$

$$\Rightarrow\ y' = \frac{8\sin(x)\sin(y)}{8\cos(x)\cos(y)}$$

$$\Rightarrow\ y' = \tan(x)\tan(y)$$

Example 6: Find the slope of the tangent line to the graph of $2x^4 = 3y^2 - y$ at the point $(1, 1)$.

$8x^3 = 6yy' - y'$

$\Rightarrow \dfrac{8x^3}{6y-1} = y'$

Plug in $(1,1)$.

$\dfrac{8 \cdot 1^3}{6 \cdot 1 - 1} = \dfrac{8}{5}$

Ex 7

Suppose $2\sin(6x + 7y) = 3xy$.

Find $\dfrac{dy}{dx}$.

$\dfrac{d}{dx}\Big(\underbrace{2\sin}_{\text{chain rule}}(\underbrace{6x + 7y}_{\text{inner function}})\Big) = \dfrac{d}{dx}(\underbrace{3x}\underbrace{y})$ product rule

$\Rightarrow 2\cos(6x+7y) \cdot (6 + 7y') = 3y + y'3x$

$\Rightarrow 12\cos(6x+7y) + 14y'\cos(6x+7y) = 3y + y'3x$

$\Rightarrow \dfrac{12\cos(6x+7y) - 3y}{3x - 14\cos(6x+7y)} = y'$

Ex 8

Suppose $\frac{5}{x} + \frac{1}{6y} = 7.$

Find $\left.\frac{dy}{dx}\right|_{(1,1/12)}.$

$$\frac{d}{dx}\Big(\underbrace{\frac{5}{x} + \frac{1}{6y}}_{5x^{-1} + \frac{1}{6}y^{-1}}\Big) = \frac{d}{dx}(7)$$

$$\Rightarrow -5x^{-2} - \frac{1}{6}y^{-2}y' = 0$$

Plug in $(1, 1/12)$.

$$-5 \cdot 1^{-2} - \frac{1}{6} \cdot \left(\frac{12}{1}\right)^2 \cdot y' = 0 \qquad y' = \frac{dy}{dx}$$

$$\Rightarrow -5 - \frac{1}{6} \cdot 144\, y' = 0$$

$$\Rightarrow -24y' = 5$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,1/12)} = -\frac{5}{24}$$