14 Lesson 14

14.1 Related Rates (Part I)

Example 1: Assume x and y are both differentiable functions of t and that $8x^5y = 16$. Find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 8$ and $x = 1$.

use implicit differentiation.

$$\frac{d}{dt}(8x^5y) = \frac{d}{dt}(16)$$

Aside: if $x=1 \Rightarrow 8 \cdot 1^5 y = 16$

$\Rightarrow y = 2$

$$\Rightarrow \underset{=1}{40x^4} \cdot \underbrace{\frac{dx}{dt}}_{8} \cdot \underset{=2}{y} + \frac{dy}{dt} \cdot 8\underset{=1}{x^5} = 0$$

$$\Rightarrow 40 \cdot 1 \cdot 8 \cdot 2 + \frac{dy}{dt} \cdot 8 = 0$$

$$\Rightarrow 640 + \frac{dy}{dt} 8 = 0$$

$$\Rightarrow 8\frac{dy}{dt} = -640$$

$$\Rightarrow \frac{dy}{dt} = -80$$

General Word Problem Approach

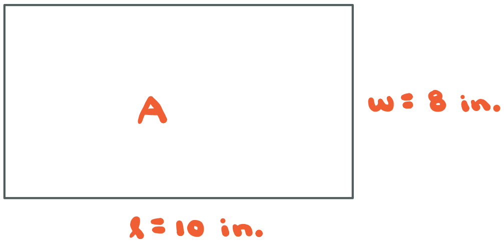
1. Read the problem, and identify any numerical values.
2. Draw a picture to represent the problem.
3. Write down an equation which describes the picture in step 2. Each variable in this equation is a function of time. Time is given by the variable t.
4. Use implicit differentiation to differentiate both sides of the equation in step 3 with respect to time (i.e. t).
5. Substitute the given information into the equation from step 4 and solve for what we want to find.

Example 2: see end of notes

$\Rightarrow$ negative

Example ~~2~~ 3: The length of a rectangle is decreasing at a rate of 3 in/s, and its width is decreasing at a rate of 2 in/s. When the length is 10 inches and the width is 8 inches, how fast is the area of the rectangle decreasing?

negative



$$A = \ell \cdot w$$

ℓ(t), w(t), A(t)

using implicit differentiation,

$$\frac{d}{dt}(A) = \frac{d}{dt}(\ell \cdot w)$$

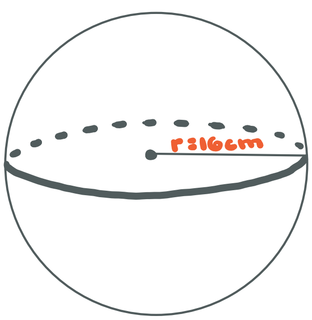
$$\Rightarrow \frac{dA}{dt} = \underbrace{\frac{d\ell}{dt}}_{-3} \cdot \underset{8}{w} + \underbrace{\frac{dw}{dt}}_{-2} \cdot \underset{10}{\ell}$$

$$\Rightarrow \frac{dA}{dt} = -24 - 20 = -44$$

So, the area of the rectangle is decreasing at a rate of 44 in^2/s.

$\Rightarrow$ negative

Example ~~3~~ 4: The radius of a spherical balloon is decreasing at a rate of 1.9 cm/s. How fast is the volume decreasing when the radius is 16 cm? Note that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.



$$V = \frac{4}{3}\pi r^3$$

r(t), V(t)

using implicit differentiation,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

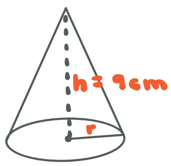
$$\Rightarrow \frac{dV}{dt} = 4\pi \underset{16}{r^2} \cdot \underbrace{\frac{dr}{dt}}_{-1.9}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi \cdot 256 \cdot (-1.9)$$

$$= -1945.6\pi$$

So, the volume of the sphere is decreasing at a rate of 1945.6π cm^3/s.

Example ~~4~~ 5: Gravel is dumped out of a truck onto the ground at 13 cm^3/s, forming a conical pile with a base diameter that is always equal to the pile's altitude (i.e. height). How fast is the altitude of the pile increasing when the pile is 9 cm high? Note that the volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.



$2r = d = h$

$\Rightarrow r = \frac{h}{2}$

$$\underset{V(+)}{V} = \frac{1}{3}\pi \underset{r(+)}{r^2} \underset{h(+)}{h}$$

$$\Rightarrow V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$$

using implicit differentiation,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right)$$

$$\Rightarrow \underbrace{\frac{dV}{dt}}_{13} = \frac{\pi}{4} \underset{9}{h}^2 \cdot \frac{dh}{dt}$$

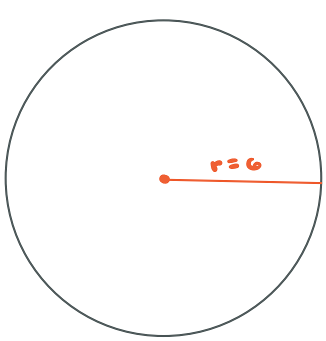
$$\Rightarrow \frac{dh}{dt} = \frac{13 \cdot 4}{81\pi} = \frac{52}{81\pi}$$

$\boxed{\text{So, the altitude of the pile is increasing at a rate of } \frac{52}{81\pi} \text{ cm/s.}}$

Ex 2

The radius of a circle is increasing at the rate of 5 ft/min.

(a) Find the rate of change of the circumference of the circle when r = 6 ft.



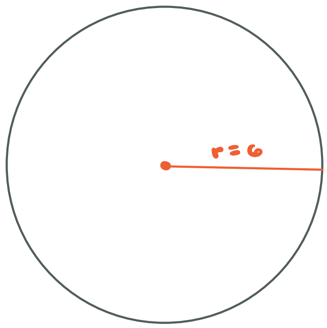
$$\underset{C(+)}{C} = 2\pi \underset{r(+)}{r}$$

using implicit differentiation,

$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$$

$$\Rightarrow \frac{dC}{dt} = 2\pi \underbrace{\frac{dr}{dt}}_{5} \Rightarrow \boxed{\frac{dC}{dt} = 10\pi \text{ ft/min}}$$

(b) Find the rate of change of the area of the circle when $r = 6$ ft.



$$\underset{A(t)}{A} = \pi \underset{r(t)}{r}^2$$

using implicit differentiation,

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dt} = 2\pi \underbrace{r}_{6} \cdot \underbrace{\frac{dr}{dt}}_{5}$$

$$\Rightarrow \boxed{\frac{dA}{dt} = 60\pi \ \text{ft}^2/\text{min}}$$