17 Lesson 17

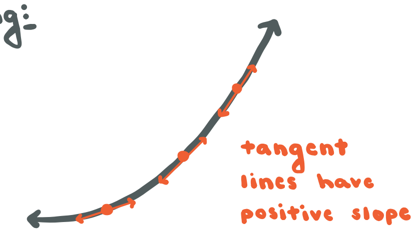
17.1 Increasing and Decreasing Functions

Last time: Critical numbers provide us with candidates for relative extrema.

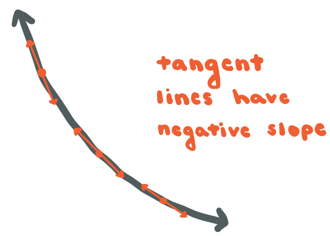
Definition: A function is increasing if $x < y$ implies $f(x) < f(y)$.

Definition: A function is decreasing if $x < y$ implies $f(x) > f(y)$.

Increasing:



Decreasing:



Theorem: Let $f(x)$ be a continuous and differentiable function on an open interval I. If $f'(x) > 0$ for all x in I, then $f(x)$ is increasing on I. If $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing on I.

Example 1: Suppose $f(x) = x^3 - 9x^2 - 21x$. Find the open intervals on which $f(x)$ is increasing or decreasing.

$$f'(x) = 3x^2 - 18x - 21$$
$$= 3(x^2 - 6x - 7)$$

$$3(x^2 - 6x - 7) = 0$$
$$\Rightarrow (x-7)(x+1) = 0$$
$$\Rightarrow x = 7, -1$$ crit. #'s

Incr.: $(-\infty, -1)$
$(7, \infty)$

Decr.: $(-1, 7)$

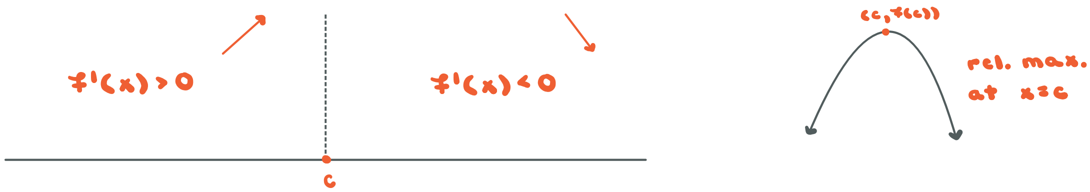
$$f'(-2) = 3((-2)^2 - 6\cdot(-2) - 7) = 3(4 + 12 - 7) > 0$$

$$f'(8) = 3(\underbrace{8^2 - \underbrace{6\cdot 8}_{48} - 7}_{64 - 55 = 9}) > 0$$

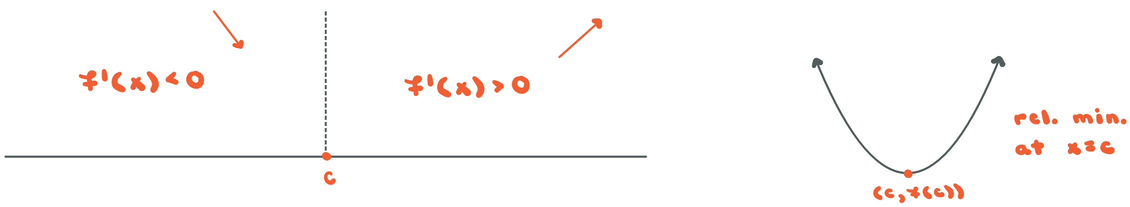
17.2 The First Derivative Test

Let c be a critical number of a function $f(x)$ that is continuous on an open interval I containing c. We consider the four following scenarios.

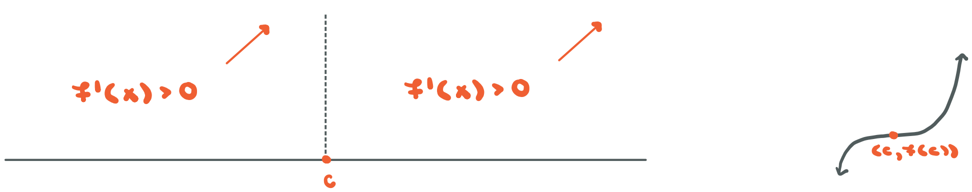
Scenario 1: $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$. This means $f(x)$ increases first and then decreases, so it reaches a relative maximum at $x = c$.



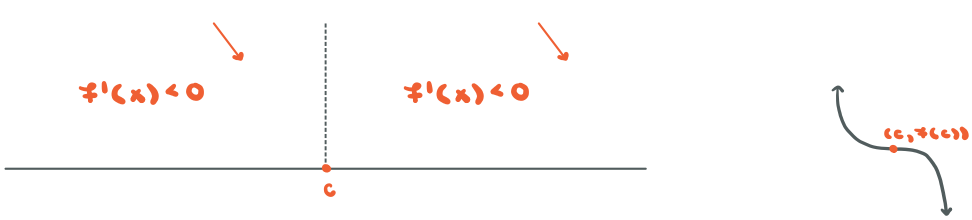
Scenario 2: $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$. This means $f(x)$ decreases first and then increases, so it reaches a relative minimum at $x = c$.



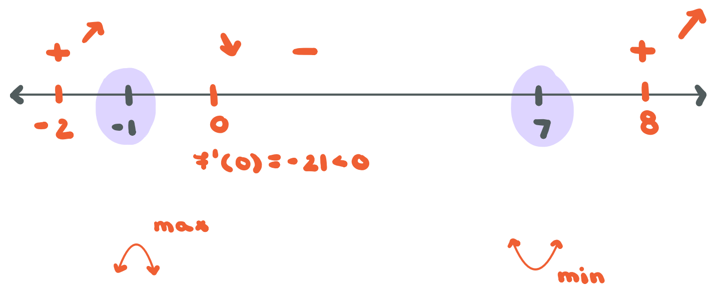
Scenario 3: $f'(x) > 0$ on both sides of $x = c$. This means $f(x)$ does not have a relative maximum or minimum at $x = c$.



Scenario 4: $f'(x) < 0$ on both sides of $x = c$. This means $f(x)$ does not have a relative maximum or minimum at $x = c$.



Example 1 (revisited): Find the relative extrema of $f(x) = x^3 - 9x^2 - 21x$.



rel. max.: $(-1, f(-1)) = (-1, 11)$

rel. min.: $(7, f(7)) = (7, -245)$

Example 2: Suppose $f(x) = 10x^2 - 4x$. Find the open intervals on which $f(x)$ is increasing or decreasing, as well as the relative extrema, if any exist.

$f'(x) = 20x - 4$

$0 = 20x - 4$

$\Rightarrow 4/20 = x$

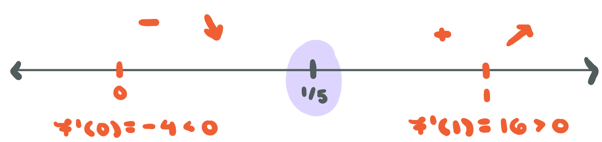
$\Rightarrow 1/5 = x$ crit. #'s

Incr.: $(1/5, \infty)$

Decr.: $(-\infty, 1/5)$

rel. min.: $(1/5, \underbrace{f(1/5)}_{=})$

$10 \cdot \left(\frac{1}{5}\right)^2 - \frac{4}{5} = \frac{2}{5} - \frac{4}{5} = -\frac{2}{5}$



Example 3: Suppose $f(x) = x^3 e^{2x+6}$. Find the open intervals on which $f(x)$ is increasing or decreasing, as well as the relative extrema, if any exist.

$f'(x) = 3x^2 e^{2x+6} + 2e^{2x+6} x^3$

$= x^2 e^{2x+6}(3 + 2x)$

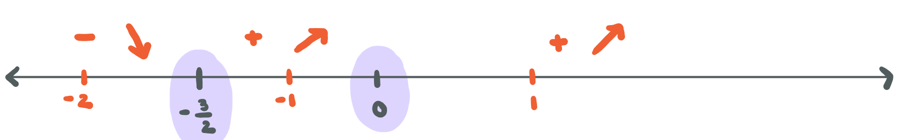
$f'(x) = 0 \Rightarrow x = 0$ or $x = -3/2$ crit. #'s

Incr.: $(-3/2, \infty)$

Decr.: $(-\infty, -3/2)$

rel. min.: $(-3/2, \underbrace{f(-3/2)}_{})$

$= -\frac{27}{8} e^3$



$f(-3/2) = \left(-\frac{3}{2}\right)^3 e^{2\left(-\frac{3}{2}\right)+6}$

$= -\frac{27}{8} e^3$

$f'(-2) < 0$

$f'(-1) > 0$

$f'(1) > 0$

Example 4: Suppose $f'(x) = e^{11x}(2x^2 - 26)$. Find the open intervals on which $f(x)$ is increasing or decreasing, as well as the x-values of the relative extrema, if any exist.

$0 = e^{11x}(2x^2 - 26)$

$\Rightarrow 0 = 2x^2 - 26$

$\Rightarrow 13 = x^2$

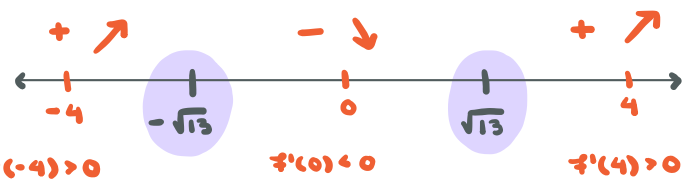
$\Rightarrow x = \pm\sqrt{13}$ crit. #'s

Incr.: $(0, -\sqrt{13}) \cup (\sqrt{13}, \infty)$

Decr.: $(-\sqrt{13}, \sqrt{13})$

rel. min. @ $x = \sqrt{13}$

rel. max. @ $x = -\sqrt{13}$



Example 5: The critical numbers of $f(x) = 5\sin(x)$ on the interval $(0, 2\pi)$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. What are the x-values on $(0, 2\pi)$ at which $f(x)$ has a relative maximum?

$f'(x) = 5\cos(x)$

$x = \frac{\pi}{2}$