18 Lesson 18

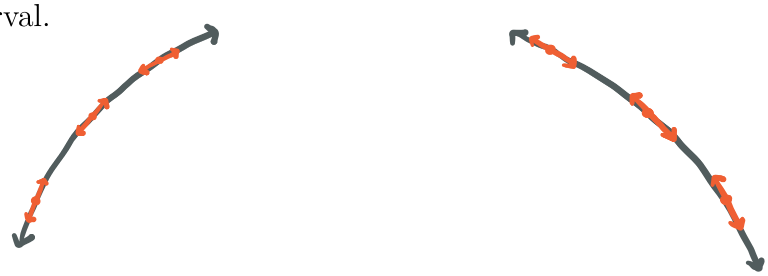
18.1 Concavity

Definition: A function $f(x)$ is concave up if it looks like either one of the figures below on an interval.

Note: The slopes are getting bigger and bigger as we move from the left to the right on the curves. In other words, the derivative of the function is increasing.

Remark: The sign of the derivative of $f(x)$ tells us whether a function is increasing or decreasing. If we know that $f''(x)$, which is the derivative of $f'(x)$, is positive, then we know that $f'(x)$ is increasing. So, $f(x)$ must be concave up.

Definition: A function $f(x)$ is concave down if it looks like either one of the figures below on an interval.



Note: The slopes are getting smaller and smaller as we move from the left to the right on the curves. In other words, the derivative of the function is decreasing.

Remark: If we know that $f''(x)$, which is the derivative of $f'(x)$, is negative, then we know that $f'(x)$ is decreasing. So, $f(x)$ must be concave down.

Theorem:

1. If $f''(x) > 0$ for all x in I, then $f(x)$ is concave up on I.
2. If $f''(x) < 0$ for all x in I, then $f(x)$ is concave down on I.

18.2 Inflection Points

Definition: $(c, f(c))$ is an inflection point if $f''(c) = 0$ or $f''(c) = \text{DNE}$, **AND** $f(x)$ changes concavity at $x = c$.

Example 1: Let $f(x) = x^4 - 8x^3 + 18x^2 - 10$. Find the open intervals on which $f(x)$ is concave up or concave down. What are the inflection points of $f(x)$? Where is the function both concave down and increasing?

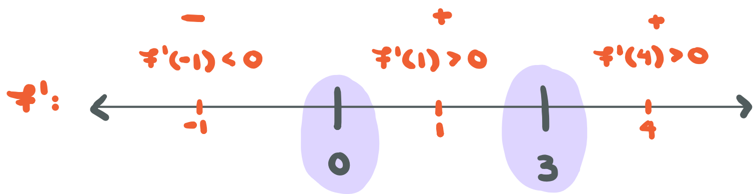
$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9) = 4x(x-3)(x-3)$

Note $f'(x) \neq$ DNE.

Solve $f'(x) = 0$.

$\Rightarrow 4x(x-3)(x-3) = 0$

$\Rightarrow x = 0, 3$ ← crit. #s



Incr: $(0, \infty)$

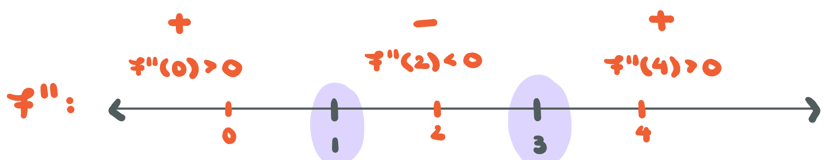
$f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4 + 3) = 12(x-3)(x-1)$

Note $f''(x) \neq$ DNE.

Solve $f''(x) = 0$.

$\Rightarrow 12(x-3)(x-1) = 0$

$\Rightarrow x = 3, 1$



Concave up: $(-\infty, 1) \cup (3, \infty)$

Concave down: $(1, 3)$

Inflection pts: $(1, f(1)) = (1, 1)$

$(3, f(3)) = (3, 17)$

Both concave down + incr.: $(0, \infty) \cap (1, 3) = (1, 3)$

18.3 Second Derivative Test

Note: Recall that the first derivative test helped us find relative extrema. The second derivative test will do the same.

Theorem: Let $f(x)$ be a function such that $f'(c) = 0$ and the second derivative of $f(x)$ exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(x)$ has a relative minimum $f(c)$ at $x = c$.
2. If $f''(c) < 0$, then $f(x)$ has a relative maximum $f(c)$ at $x = c$.
3. If $f''(c) = 0$, the second derivative test is inconclusive. We need to use the first derivative test to check for potential relative extrema.

Example 2: Let $f(x) = x^4 - 4x^3 + 4$. Use the second derivative test to find any relative extrema of $f(x)$.

$f'(x) = 4x^3 - 12x^2$
$\quad = 4x^2(x-3)$

Note $f'(x) \neq$ DNE

Solve $f'(x) = 0$

$0 = 4x^2(x-3)$

$\Rightarrow x = 0, 3$

$f''(x) = 12x^2 - 24x$

plug in $x = 0, 3$

$f''(0) = 0 \Rightarrow$ inconclusive

$f''(3) = 12 \cdot 9 - 24 \cdot 3 > 0 \Rightarrow$ rel. min. at $(3, f(3)) = (3, -23)$

f': $-$ ↘ $f'(-1) < 0$ | -1 | 0 | $-$ ↘ $f'(1) < 0$ | 3

$\Rightarrow$ no rel. extrema at $x = 0$

Example 3: Let $f(x) = 9\ln(x^2+1)$. Find the open intervals on which $f(x)$ is concave up or concave down. What are the inflection points of $f(x)$?

$$f'(x) = \frac{9}{x^2+1} \cdot 2x = \frac{18x}{x^2+1}$$

$$f''(x) = \frac{18(x^2+1) - 2x \cdot 18x}{(x^2+1)^2}$$

$$= \frac{18x^2+18-36x^2}{(x^2+1)^2}$$

$$= \frac{18-18x^2}{(x^2+1)^2}$$

Note $f''(x) \neq$ DNE

Solve $f''(x) = 0$.

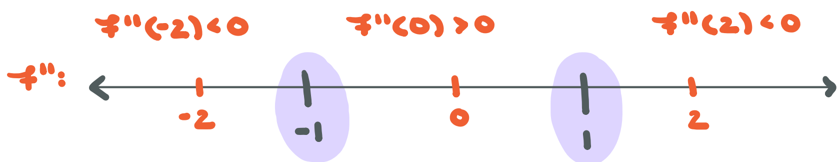
$\Rightarrow \; 0 = 18 - 18x^2$

$\Rightarrow \; 0 = 18(1-x^2)$

$\Rightarrow \; 0 = 1 - x^2$

$\Rightarrow \; x^2 = 1$

$\Rightarrow \; x = \pm 1$



Concave up: $(-1,1)$

Concave down: $(-\infty,-1) \cup (1,\infty)$

Inflection pts: $(-1, f(-1)) = (-1, 9\ln(2))$

$(1, f(1)) = (1, 9\ln(2))$

Example 4: Let $f(x) = x^5 - 5x^4 + 5x^3$. Use the second derivative test to find any relative extrema of $f(x)$.

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$\quad = 5x^2(x^2 - 4x + 3)$

$\quad = 5x^2(x-3)(x-1)$

Note $f'(x) \neq$ DNE.

Solve $f'(x) = 0$.

$0 = 5x^2(x-3)(x-1)$

$\Rightarrow x = 0, 3, 1$

$f''(x) = 20x^3 - 60x^2 + 30x$

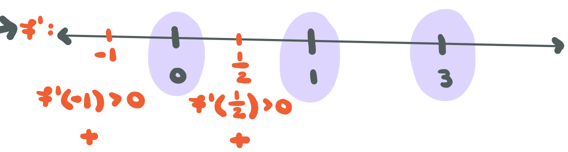
$\quad = 10x(2x^2 - 6x + 3)$

Plug in $x = 0, 3, 1$.

$f''(0) = 0 \Rightarrow$ inconclusive

$f''(3) > 0 \Rightarrow$ rel. min.

$f''(1) < 0 \Rightarrow$ rel. max.



$\Rightarrow$ no rel. extrema at $x=0$

Rel. min.: $(3, \overbrace{f(3)}^{-27})$

Rel. max.: $(1, \underbrace{f(1)}_{1})$