19 Lesson 19

19.1 Absolute Extrema on an Interval

Definition: An absolute maximum is the largest function value on the entire interval.

Definition: An absolute minimum is the smallest function value on the entire interval.

19.2 Absolute Extrema on a Closed Interval

Remark 1: If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ has both an absolute maximum and an absolute minimum on the interval.

Remark 2: The absolute extrema only occur either at the critical numbers or at the end points (for a continuous function on a closed interval).

Steps to Find Absolute Extrema (on Closed Interval)

1. Find the critical numbers of $f(x)$.
2. Evaluate $f(x)$ at the critical numbers on $[a, b]$ and the end points.
3. Compare the results from step 2 to determine the absolute extrema.

Example 1: Find the absolute extrema of $f(x) = xe^{-x} + 2$ on the closed interval $[0, 4]$.

$f'(x) = e^{-x} - xe^{-x}$

$= e^{-x}(1-x)$

$0 = \underbrace{e^{-x}}_{\text{never } 0}(1-x)$

$\Rightarrow \quad x = 1$

x	$f(x)$
0	2
1	$\frac{1}{e} + 2 \approx 2.37$
4	$\frac{4}{e^4} + 2 \approx 2.07$

abs. max. : $(1, \frac{1}{e} + 2)$

abs. min. : $(0, 2)$

19.3 Absolute Extrema on an Open or Half-Open and Half-Closed Interval with only One Critical Number

Remark 3: If a relative maximum occurs at the critical number, then the absolute maximum also occurs at this critical number. We cannot say anything about an absolute minimum.

Remark 4: If a relative minimum occurs at the critical number, then the absolute minimum also occurs at this critical number. We cannot say anything about an absolute maximum.

Note: In the scenarios in remark 3 and 4, use the first or second derivative test to determine whether the critical number produces a maximum or minimum.

Example 2: Find the absolute maximum of $y = -x^2 + 10$ on the interval $(-2, 2)$.

$y' = -2x$

$0 = -2x$

$\Rightarrow x = 0$

First Deriv. Test

$y'(x)$ sign chart: -1, 0, 1; $+$ ↗ on the left of 0, ↘ $-$ on the right of 0

abs. max.

Second Deriv. Test

$y'' = -2$

$y''(0) = -2 \Rightarrow$ abs. max.

abs. max. : $(0, 10)$

Example 3: Find the absolute minimum of $y = \frac{3x^2}{x+1}$ on the interval $(-1, 5]$.

$$y' = \frac{\overbrace{6x(x+1) - 1 \cdot 3x^2}^{3x^2+6x}}{(x+1)^2}$$

Note $y' =$ DNE when $x = -1$ but y not defined when $x = -1$.

$0 = 3x^2 + 6x$

$= 3x(x+2)$

$\Rightarrow x = 0, -2$

↑ not in interval

First Deriv. Test

$y'(x)$ sign chart: -2, $-\frac{1}{2}$, 0, 1; $-$ ↘ on the left of 0, ↗ $+$ on the right of 0

abs. min. : $(0, 0)$

Practice Problems

1 $y = \frac{1}{3}x^3 - 8x + 6$

Find x-values where abs. min. or max. of y occur on $[-1, 6]$.

$$y' = x^2 - 8$$

$$0 = x^2 - 8$$
$$\Rightarrow 8 = x^2$$
$$\Rightarrow \pm\sqrt{8} = x$$
$$\Rightarrow \pm 2\sqrt{2} = x$$

Note $-2\sqrt{2}$ is not in $[-1, 6]$.

x	y
-1	$-\frac{1}{3} + 8 + 6 = 14 - \frac{1}{3} \approx 13.66$
$2\sqrt{2}$	$\frac{8}{3}(\sqrt{2})^3 - 16\sqrt{2} + 6 = \frac{16}{3}\sqrt{2} - 16\sqrt{2} + 6 \approx -9.085$
6	$\underbrace{\frac{1}{3}\cdot 6^3 - 8\cdot 6 + 6}_{6(\underbrace{6^2/3 - 8 + 1}_{12 - 8 + 1})} = 30$

abs. min. at $x = 2\sqrt{2}$

abs. max. at $x = 6$

2 Find the abs. max. of

$$f(x) = \frac{x}{x^2 + 4}$$

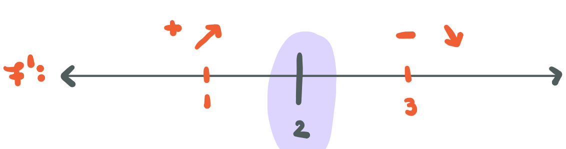
on $(0, 10)$.

$$f'(x) = \frac{1\cdot(x^2+4) - 2x\cdot x}{(x^2+4)^2}$$
$$= \frac{x^2 + 4 - 2x^2}{(x^2+4)^2}$$
$$= \frac{4 - x^2}{(x^2+4)^2}$$

Note $f'(x) \neq$ DNE.

$$0 = 4 - x^2$$
$$\Rightarrow 4 = x^2$$
$$\Rightarrow x = \pm 2$$

Note -2 is not in $(0, 10)$.



$$f(2) = \frac{2}{2^2 + 4} = \frac{2}{8} = \frac{1}{4}$$

abs. max.: $(2, 1/4)$