21 Lesson 21

21.1 Limits at Infinity

Example 1: Find the following limits.

(a)

$$\lim_{x\to\infty} \frac{7}{x} = 0$$

x	5	50	500	5,000
$f(x)$	$7/5$	$7/50$	$7/500$	$7/5000$

(b)

$$\lim_{x\to-\infty} \left(\frac{x}{7}+3\right) = -\infty$$

(c)

$$\lim_{x\to-\infty} \left(\underbrace{\frac{x}{10}}_{\text{goes to } -\infty} - \underbrace{\frac{1}{x}}_{\text{goes to } 0}\right) = -\infty$$

Note: Finding the limit of a rational function at $\pm\infty$ is the same as taking the limit of the ratio of its leading terms (the terms with the highest power of x in the numerator and denominator).

Example 2: Find the following limits.

(a)

$$\lim_{x\to\infty} \frac{3x^2+4}{6x^2-3x} = \lim_{x\to\infty} \frac{3x^2}{6x^2}$$

$$= \lim_{x\to\infty} \frac{1}{2}$$

$$= \frac{1}{2}$$

(b)

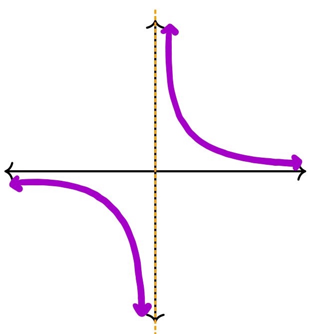
$$\lim_{x \to -\infty} \frac{x+18}{x^2+2x+1} = \lim_{x \to -\infty} \frac{x}{x^2}$$

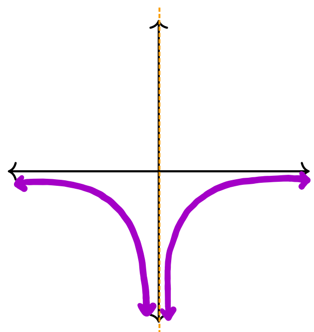
$$= \lim_{x \to -\infty} \frac{1}{x}$$

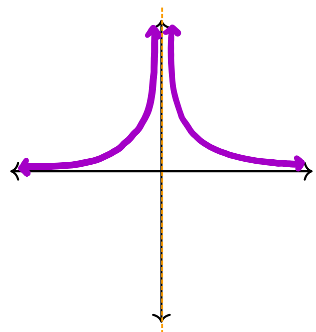
$$= 0$$

21.2 Vertical Asymptotes

Recall from lesson 4 that a function $f(x)$ has a vertical asymptote at $x = c$ if $\lim_{x \to c^-} f(x) = \pm\infty$ and/or $\lim_{x \to c^+} f(x) = \pm\infty$. For rational functions, we only need to check the values for x that make the denominator zero (and the numerator nonzero) when looking for vertical asymptotes.



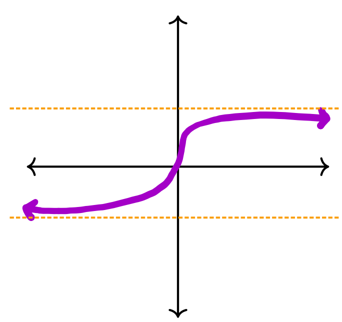




21.3 Horizontal Asymptotes

The line $y = L$, where L is a constant, is a horizontal asymptote of the graph of $f(x)$ if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$



Example 3: Find any horizontal asymptotes for the following functions.

(a) $f(x) = x^2 + 1$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

none

(b) $g(x) = \frac{2x^3+1}{x^4}$

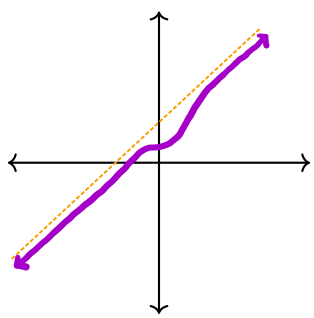
$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{2x^3}{x^4} = \lim_{x \to \infty} \frac{2}{x} = 0$

$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{2x^3}{x^4} = \lim_{x \to -\infty} \frac{2}{x} = 0$

$y = 0$

21.4 Slant Asymptotes

A slant asymptote is an asymptote for a function $f(x)$ which is not vertical nor horizontal; it is a line of the form $y = mx + b$, where m and b are constants.



Note: We use polynomial long division to find slant asymptotes. The quotient (the answer ignoring any remainder) is the slant asymptote. A rational function will have a slant asymptote only when the degree of the numerator is one higher than the degree of the denominator.

Example 4: Find the slant asymptote for the following function.

$$\frac{x^3 + 2x^2 + 1}{x^2 + 1}$$

$$\begin{array}{r l}
 & x + 2 \\
x^2 + 1 \,\big)\!\! & x^3 + 2x^2 + 0x + 1 \\
 & -(x^3 + x) \\
\hline
 & 2x^2 - x + 1 \\
 & -(2x^2 + 2) \\
\hline
 & -x - 1
\end{array}$$

$y = x + 2$

21.5 Practice Problems

Example 5: Find any vertical, horizontal, and slant asymptotes for the following function.

$$h(x) = \frac{-2x^2}{x^2 - 9}$$

vertical: Note denominator is zero when $x = \pm 3$

$\lim_{x \to 3} h(x) \rightsquigarrow \frac{-2 \cdot 3^2}{3^2 - 9} = \frac{-18}{0}$ (case 2 limit) $\Rightarrow$ V.A. at $x = 3$

↳ see lesson 4

$\lim_{x \to -3} h(x) \rightsquigarrow \frac{-2 \cdot (-3)^2}{(-3)^2 - 9} = \frac{-18}{0}$ (case 2 limit) $\Rightarrow$ V.A. at $x = -3$

↳ see lesson 4

horizontal:

$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{-2x^2}{x^2} = \lim_{x \to -\infty} -2 = -2$

$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{-2x^2}{x^2} = \lim_{x \to \infty} -2 = -2$

H.A. at $y = -2$

slant: degrees of numerator and denominator same $\Rightarrow$ no S.A.