22 Lesson 22

22.1 A Summary of Curve Sketching

Idea: Limits and derivatives are tools we can use to better understand what the graph of a function looks like. We have used these tools in the last few weeks to find asymptotes, extrema, changes in concavity, etc. of a function. Today, we will use all of this information to help us sketch the graph of a function.

22.2 Strategy

Let $f(x)$ be a function.

1. Find any $x-$ and $y-$ intercepts of $f(x)$.
2. Find the intervals on which $f(x)$ is increasing and decreasing.
3. Find any relative extrema of $f(x)$.
4. Find the intervals on which $f(x)$ is concave up and concave down.
5. Find any inflection points of $f(x)$.
6. Find any asymptotes of $f(x)$.
7. Use the information from steps 1-6 to sketch a graph of the function.

22.3 Practice Problems

Example 1: Use the strategy above to sketch a graph of $f(x) = \frac{2x^2}{x+1}$.

Example 2: Use the following information to sketch a graph of $f(x)$.

- The point $(-2, 0)$ is on the graph of $f(x)$.
- $f(x)$ has a horizontal asymptote at $y = 0$, vertical asymptotes at $x = \pm 1$, and no slant asymptotes.
- $f'(x) > 0$ on the intervals $(-\infty, -1)$ and $(-1, 0)$.
- $f'(x) < 0$ on the intervals $(0, 1)$ and $(1, \infty)$.
- $f''(x) > 0$ on the intervals $(-\infty, -1)$ and $(1, \infty)$.
- $f''(x) < 0$ on the interval $(-1, 1)$.

On $(-\infty,-1)$: increasing + concave up

On $(1,\infty)$: decreasing + concave up

On $(-1,0)$: increasing

On $(0,1)$: decreasing

On $(-1,1)$: concave down

Example 1: $f(x) = \dfrac{2x^2}{x+1}$

$$f'(x) = \frac{4x(x+1) - 1\cdot 2x^2}{(x+1)^2} = \frac{2x^2+4x}{(x+1)^2}$$

$$f''(x) = \frac{(4x+4)(x+1)^2 - 2(x+1)(2x^2+4x)}{(x+1)^4}$$

$$= \frac{(4x+4)(x+1) - 2(2x^2+4x)}{(x+1)^3}$$

$$= \frac{4x^2+8x+4-4x^2-8x}{(x+1)^3}$$

$$= \frac{4}{(x+1)^3}$$

x-int: Set $f(x) = 0$.

$0 = \dfrac{2x^2}{x+1} \Rightarrow 2x^2 = 0 \Rightarrow x = 0 \rightsquigarrow (0,0)$ on graph

y-int: Let $x = 0$ in $f(x)$.

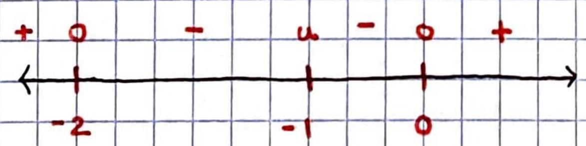
$f(0) = \dfrac{0}{1} = 0 \rightsquigarrow (0,0)$ on graph

Inc./Decr.: Note $f'(x)$ is undefined at $x=-1$. But, $x=-1$ not in domain of $f(x)$.

Set $f'(x) = 0$.

$0 = \dfrac{2x^2+4x}{(x+1)^2} \Rightarrow 2x^2+4x = 0 \Rightarrow 2x(x+2) = 0 \Rightarrow x = 0, -2$

sign chart:



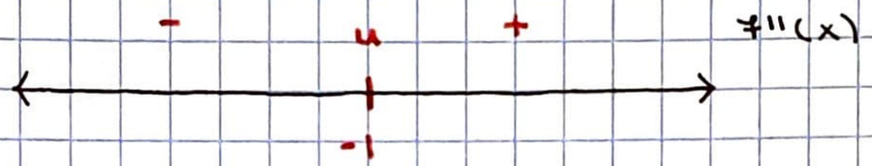
Inc.: $(-\infty,-2) \cup (0,\infty)$

Decr.: $(-2,-1) \cup (-1,0)$

C.U./C.D.: Note $f''(x)$ is undefined at $x=-1$, but $x=-1$ not in domain of $f(x)$. So, $x=-1$ is not I.P.

Note $f''(x)$ is never 0.

sign chart:



C.U.: $(-1, \infty)$
C.D.: $(-\infty, -1)$

Asymptotes

V.A.: Denominator of $f(x)$ is zero when $x=-1$. Notice we can't cancel a factor of $x+1$ in the numerator. Also, $f(-1) = \frac{2}{0}$. So, we have a V.A. at $x=-1$.

H.A.: $$\lim_{x\to\infty} \frac{2x^2}{x+1} = \lim_{x\to\infty} \frac{2x^2}{x} = \lim_{x\to\infty} 2x = \infty$$

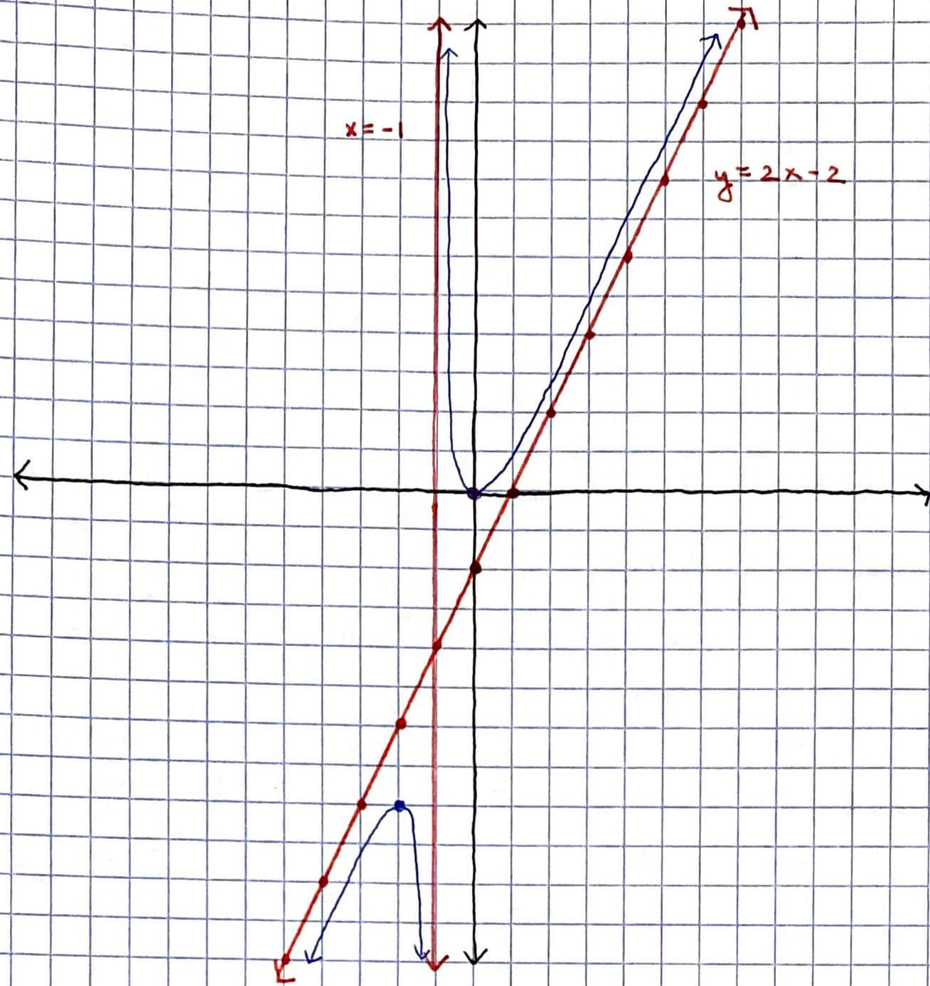
$$\lim_{x\to-\infty} \frac{2x^2}{x+1} = \lim_{x\to-\infty} \frac{2x^2}{x} = \lim_{x\to-\infty} 2x = -\infty$$

So, no H.A.

S.A.: Use polynomial long division.

$$\begin{array}{r|l} & 2x - 2 \\ \hline x+1 & 2x^2 + 0x + 0 \\ & -(2x^2 + 2x) \\ \hline & -2x + 0 \\ & -(-2x - 2) \\ \hline & 2 \end{array}$$

So, S.A. at $y = 2x-2$. Note $\deg(2x^2) = 2 = 1+1 = \deg(x)+1$, i.e. we knew we must have a S.A.



Example 2: