23 Lesson 23

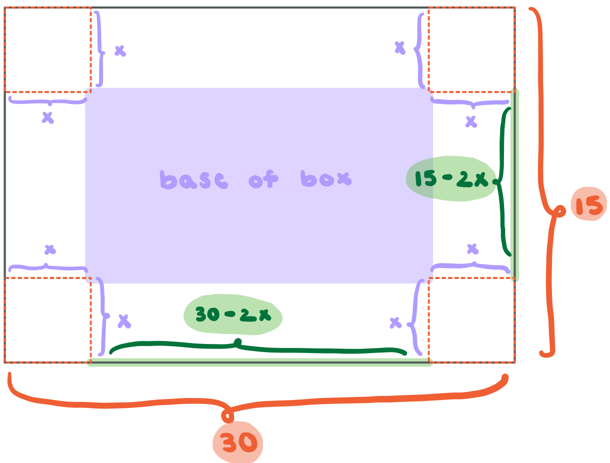
23.1 Optimization

Idea: Find absolute maximums or minimums of functions which model word problems.

Strategy

1. Draw a picture to represent the problem. Include variable names in this picture.
2. Identify the objective function, i.e. the function we want to minimize or maximize. Use the variable names from step 1.
3. Identify the constraint equation(s), i.e. the equation(s) which describe the constraints on the variables. Use the variable names from step 1.
4. If applicable, use the constraint equation(s) to rewrite the objective function in terms of one variable.
5. Identify the interval of interest of the objective function, i.e. the interval of valid inputs into the objective function.
6. Find the absolute extrema (if any exist) of the objective function on the interval of interest. If applicable, check the endpoints of the interval of interest.

Example 1: A piece of cardboard is ~~12~~ 15 inches by ~~18~~ 30 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What size should the squares be cut to maximize the volume of the resulting box?



objective function

$h = x$

$V = \ell \cdot w \cdot h$

$\ell = 15 - 2x$

$w = 30 - 2x$

So, $V = (15-2x)(30-2x)x$

$= (450 - 30x - 60x + 4x^2)x$

$= 4x^3 - 90x^2 + 450x$

Interval of Interest

B/c x is length $\Rightarrow x > 0$

Note $x \neq 0$, as then we wouldn't be able to fold into box.

Does x have a max?

Note $0 < \ell < 15$ and $0 < w < 30$.

$0 < 15 - 2x$ $\Rightarrow 2x < 15$ $\Rightarrow x < 15/2$

$0 < 30 - 2x$ $\Rightarrow 2x < 30$ $\Rightarrow x < 15$

So, x is in $(0, 15/2)$, i.e. $0 < x < 15/2$.

Now, $V' = 12x^2 - 180x + 450$

$= 6(2x^2 - 30x + 75)$

Note $V' \not\equiv$ DNE. Also, $V' = 0$

$\Rightarrow 2x^2 - 30x + 75 = 0$

$\Rightarrow x = \dfrac{-(-30) \pm \sqrt{(-30)^2 - 4 \cdot 2 \cdot 75}}{2 \cdot 2}$ (quadratic formula)

$= \dfrac{30 \pm \sqrt{900 - 600}}{4}$

$= \dfrac{30 \pm \sqrt{3 \cdot 100}}{4}$ ($\rightarrow 10\sqrt{3}$)

$= \dfrac{15 \pm 5\sqrt{3}}{2}$

Note $\dfrac{15 + 5\sqrt{3}}{2} > \dfrac{15}{2}$ $\Rightarrow$ not in interval

$\dfrac{15 - 5\sqrt{3}}{2} \approx 3.17$ $\Rightarrow$ in interval

Need to make sure V has a max. when $x = \dfrac{15 - 5\sqrt{3}}{2}$.

$V'' = 24x - 180$

Note $V''(x) < 0$ when $x = \dfrac{15 - 5\sqrt{3}}{2}$ ← square dims.

$\Rightarrow$ max!

Example 2: Find the dimensions of the rectangle, with perimeter 36, that has the largest area.

objective function: $A = xy$

Constraint: $2x + 2y = 36$

↳ $x + y = 18$

$\Rightarrow y = 18 - x$

So, $A = xy = x(18-x) = 18x - x^2$

Interval of Interest:

Note $x > 0$.

Also, $y > 0 \Rightarrow 18 - x > 0 \Rightarrow 18 > x$

So, $0 < x < 18$.

Note $A' = 18 - 2x$.

$A' \neq$ DNE.

Solve $A' = 0 \Rightarrow 18 - 2x = 0$

$\Rightarrow x = 9$

in interval! check that A has max. at $x = 9$

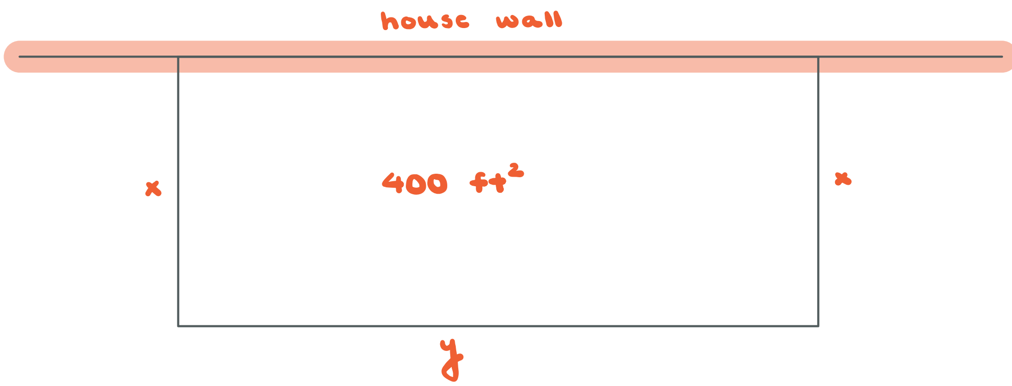
↳ $A'' = -2 < 0$

$\Rightarrow$ max!

Dim: $x = 9$

$y = 18 - x = 18 - 9 = 9$

Example 3: We want to fence a rectangular garden alongside a wall of the house. Find the least amount of fencing needed so that the area of the garden is exactly 400 ft^2. Note that one side of the garden is protected by the house wall.



objective function: $P = 2x + y$

constraint: $400 = xy \Rightarrow y = \frac{400}{x}$

So, $P = 2x + y = 2x + \frac{400}{x}$ (*)

Interval of Interest: B/c x length $\Rightarrow x > 0$

Also, x has no max.

So, $0 < x < \infty$.

Now, $P' = 2 - 400x^{-2}$

$= 2 - \frac{400}{x^2}$

Note $P' =$ DNE when $x = 0$ but 0 is not in $(0, \infty)$

Solve $P' = 0$

$2 - \frac{400}{x^2} = 0$

$\Rightarrow 2 = \frac{400}{x^2}$

$\Rightarrow 2x^2 = 400$

$\Rightarrow x^2 = 200$

$\Rightarrow x = \pm\sqrt{200}$ ($= \sqrt{2 \cdot 100}$)

$= \pm 10\sqrt{2}$

Note $-10\sqrt{2}$ not in $(0, \infty)$.

Check that P has min. at $x = 10\sqrt{2}$

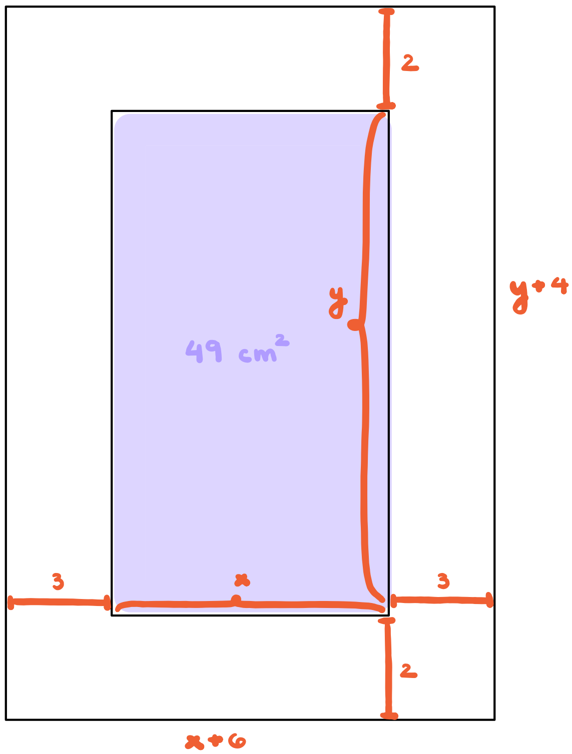
↳ $P'' = \frac{800}{x^3} > 0$ at $x = 10\sqrt{2} \Rightarrow$ min!

So,

$P(10\sqrt{2}) \overset{(*)}{=} 20\sqrt{2} + \frac{400}{10\sqrt{2}}$ ($\frac{40}{\sqrt{2}} = \frac{40\sqrt{2}}{2} = 20\sqrt{2}$)

$= 40\sqrt{2}$ feet

Example 4: We are designing a poster. We want the poster to contain a printing section in the middle with an area of 49 cm^2. We also want the poster to have 2 cm margins at the top and bottom and 3 cm margins on the sides. Find the dimensions of such a poster which has the smallest possible area.



objective function: $A = (y+4)(x+6)$

Constraint: $xy = 49 \Rightarrow y = 49/x$

So, $A = (49/x + 4)(x+6)$

$= 49 + 294/x + 4x + 24$

$= 294/x + 4x + 73$

Interval of Interest: B/c x length $\Rightarrow x > 0$

Also, x has no max.

So, $0 < x < \infty$.

Now, $A' = -294x^{-2} + 4$

Note $A' =$ DNE when $x = 0$ but 0 is not in $(0, \infty)$

Solve $A' = 0$

$-294x^{-2} + 4 = 0$

$\Rightarrow 4 = \frac{294}{x^2}$

$\Rightarrow 4x^2 = 294$

$\Rightarrow x^2 = 73.5$

$\Rightarrow x = \pm\sqrt{73.5}$

Note $-\sqrt{73.5}$ is not in $(0, \infty)$.

Check that A has a min. at $x = \sqrt{73.5}$.

$A'' = 588x^{-3} > 0$ for all x ✓

Dims: $y+4$ by $x+6$

$y = \frac{49}{x}$, so $\frac{49}{\sqrt{73.5}} + 4$ cm by $\sqrt{73.5} + 6$ cm