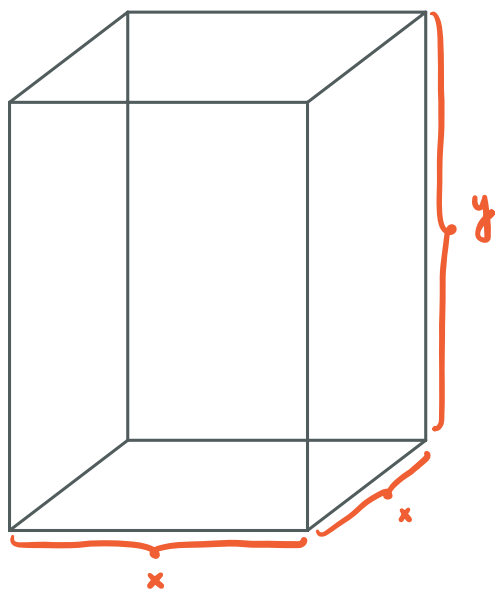


24 Lesson 24

24.1 Optimization

Example: A box with a square base and no top is to be built with a volume of 2000 in^3 . Find the dimensions of the box so that we use the least amount of material possible.

minimize surface area



Obj. function: $S = x^2 + 4xy$

constraint: $2000 = x^2y$
 $\hookrightarrow y = \frac{2000}{x^2}$

So, $S = x^2 + 4x \cdot \frac{2000}{x^2} = x^2 + \frac{8000}{x}$

Interval: Note $x > 0$.

Also, $y > 0 \Rightarrow \frac{2000}{x^2} > 0$. True for all $x > 0$, so no new info. for max. of x .

So, x is in $(0, \infty)$.

Find min. of S on $(0, \infty)$.

$$S' = 2x - \frac{8000}{x^2}$$

S' is DNE when $x=0$ but 0 is not in $(0, \infty)$.

Solve $S' = 0$.

$$\hookrightarrow 2x - \frac{8000}{x^2} = 0$$

$$\Rightarrow 2x = \frac{8000}{x^2}$$

$$\Rightarrow 2x^3 = 8000$$

$$\Rightarrow x^3 = 4000$$

$$\Rightarrow x = \sqrt[3]{4000}$$
$$= \sqrt[3]{1000 \cdot 4}$$
$$= 10\sqrt[3]{4}$$

← verify min.
using 2nd
derivative test

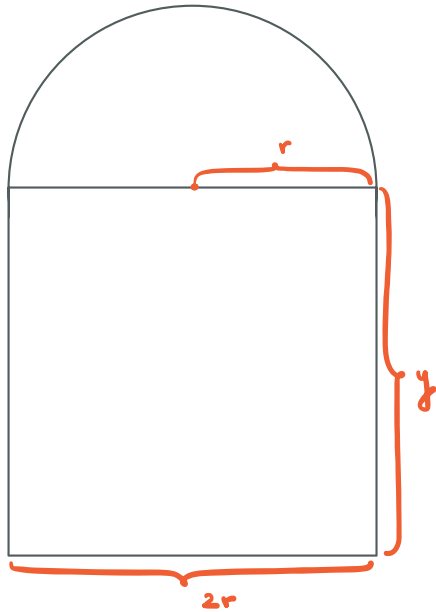
$$S'' = 2 + \frac{16000}{x^3}$$

$$S''(10\sqrt[3]{4}) > 0 \Rightarrow S \text{ has abs. min. at } x = 10\sqrt[3]{4}$$

$$x = 10\sqrt[3]{4} \text{ in.}$$

$$y = \frac{2000}{(10\sqrt[3]{4})^2} = \frac{2000}{100 \sqrt[3]{16}} = \frac{20}{2\sqrt[3]{2}} = \frac{10}{\sqrt[3]{2}} \text{ in.}$$

Example: A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 28 feet.



Obj. function: $A = \frac{\pi r^2}{2} + 2ry$

constraint: $28 = \pi r + 2y + 2r$
 $\hookrightarrow \frac{28 - \pi r - 2r}{2} = y$

So, $A = \frac{\pi r^2}{2} + 2r \left[\frac{28 - \pi r - 2r}{2} \right]$
 $= \frac{\pi r^2}{2} + 28r - \pi r^2 - 2r^2$

Interval: Note $r > 0$.

Also, $y \geq 0 \Rightarrow \frac{28 - \pi r - 2r}{2} \geq 0$

$\Rightarrow 28 - \pi r - 2r \geq 0$

$\Rightarrow 28 - r(\pi + 2) \geq 0$

$\Rightarrow -r(\pi + 2) \geq -28$

$\Rightarrow r \leq \frac{28}{\pi + 2}$

So, x is in $(0, 28/(\pi + 2)]$.

Find max. of A on $(0, 28/(\pi + 2)]$.

$A' = \pi r + 28 - 2\pi r - 4r$
 $= 28 - 4r - \pi r$
 $= 28 - (4 + \pi)r$

Note $A' \neq \text{DNE}$.

Solve $A' = 0$.

$\hookrightarrow 28 - (4 + \pi)r = 0$

$\Rightarrow r = \frac{28}{4 + \pi}$ ← verify max.

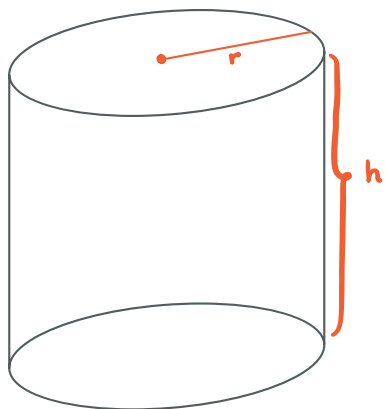
$A'' = -(4 + \pi)$

$\Rightarrow A''(28/(4 + \pi)) < 0 \Rightarrow \text{abs. max.}$

$r = \frac{28}{4 + \pi}$ ft.

$y = \frac{28 - \frac{28\pi}{4 + \pi} - \frac{56}{4 + \pi}}{2} = 14 - \frac{14\pi}{4 + \pi} - \frac{28}{4 + \pi}$
mult. by $\frac{4 + \pi}{4 + \pi}$
 $= \frac{56 + 14\pi - 14\pi - 28}{4 + \pi}$
 $= \frac{28}{4 + \pi}$ ft.

Example: Suppose that a cylindrical can is able to hold 128π in³ of liquid. Find the dimensions of the can, namely the radius r and height h , that will minimize the amount of material used. Note that the volume of a cylinder is $\pi r^2 h$ and the surface area is $2\pi r h + 2\pi r^2$, where r is the radius and h is the height of the cylinder.



Obj. function: $S = 2\pi r h + 2\pi r^2$

constraint: $128\pi = \pi r^2 h$
 $\hookrightarrow 128 = r^2 h$
 $\Rightarrow \frac{128}{r^2} = h$

So, $S = 2\pi r \cdot \frac{128}{r^2} + 2\pi r^2$
 $= \frac{256\pi}{r} + 2\pi r^2$

Interval: Note $r > 0$.

Also, $h > 0 \Rightarrow \frac{128}{r^2} > 0$. True for all $r > 0$, so no

new info. for max. of r .

So, r is in $(0, \infty)$.

Find min. of S on $(0, \infty)$.

$$S' = \frac{-256\pi}{r^2} + 4\pi r$$

S' is DNE when $r=0$ but 0 is not in $(0, \infty)$.

Solve $S' = 0$.

$$\hookrightarrow \frac{-256\pi}{r^2} + 4\pi r = 0$$

$$\Rightarrow 4\pi r = \frac{256\pi}{r^2}$$

$$\Rightarrow r^3 = \frac{256\pi}{4\pi}$$

$$\Rightarrow r^3 = 64$$

$$\Rightarrow r = 4 \leftarrow \text{verify min.}$$

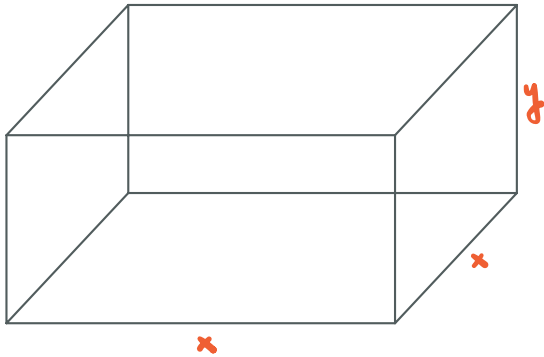
$$S'' = \frac{512\pi}{r^3} + 4\pi$$

$$S''(4) > 0 \Rightarrow \text{abs. min.}$$

$$r = 4 \text{ in.}$$

$$h = \frac{128}{4^2} = 8 \text{ in.}$$

Example: A rectangular box has a square base. The sum of the two base side lengths and the height is 30 in. What is the maximum possible volume?



Obj. function: $V = x^2 y$

constraint: $2x + y = 30$
 $\hookrightarrow y = 30 - 2x$

So, $V = x^2(30 - 2x) = 30x^2 - 2x^3$

Interval: Note $x > 0$.

Also, $y > 0 \Rightarrow 30 - 2x > 0 \Rightarrow 30 > 2x \Rightarrow 15 > x$

So, x is in $[0, 15]$.

Find max. of V on $[0, 15]$.

$V' = 60x - 6x^2$

Note $V' \neq \text{DNE}$.

Solve $V' = 0$.

$\hookrightarrow 0 = 60x - 6x^2$
 $= 6x(10 - x)$
 $\Rightarrow x = 0, 10$

x	V
0	0
10	$3000 - 2000 = 1000$
15	0

\Rightarrow abs. max. at $x = 10$

So, $y = 30 - 2x = 30 - 20 = 10$

$V = x^2 y = 10^3 = 1000 \text{ in}^3$