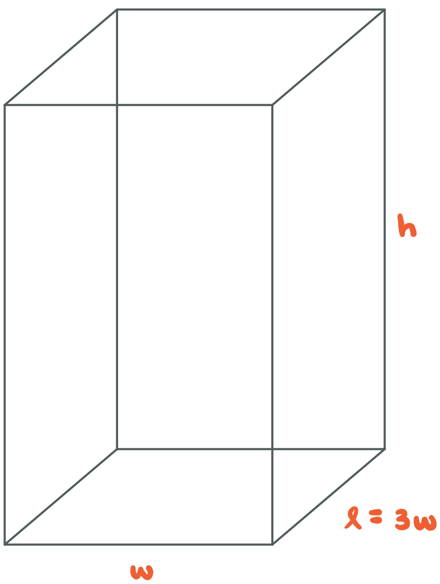
25 Lesson 25

25.1 Optimization

Example: A rectangular box is to be constructed with a volume of 500 in^3. The material for the sides costs \$2 per square inch, and the material for the top and bottom (the two bases) costs \$1 per square inch. Suppose that the length of each base is 3 times the width of the base. Determine the minimum cost to construct this box.



Obj.: $C = 2(2\cdot 3wh + 2\cdot wh) + 1\cdot(2\cdot 3w^2)$
$= 12wh + 4wh + 6w^2$
$= 16wh + 6w^2$

Constraint: $500 = \ell wh = (3w)wh = 3w^2h$
$\hookrightarrow \frac{500}{3w^2} = h$

So, $C = 16w\cdot\left[\frac{500}{3w^2}\right] + 6w^2$
$= \frac{8000}{3w} + 6w^2$

Interval: Note $0 < w < \infty$.

$C' = -\frac{8000}{3w^2} + 12w$

Note $C' = \text{DNE}$ when $w = 0$, but 0 is not in $(0, \infty)$.

Solve $C' = 0$.
$\hookrightarrow -\frac{8000}{3w^2} + 12w = 0$
$\Rightarrow 12w = \frac{8000}{3w^2}$
$\Rightarrow w^3 = \frac{8000}{36}$
$\Rightarrow w = \frac{\sqrt[3]{8000}}{\sqrt[3]{36}}$
$= \frac{20}{\sqrt[3]{36}}$ ← verify this is min. of C on $(0,\infty)$

$C'' = \frac{16000}{3w^3} + 12$

$C''\left(\frac{20}{\sqrt[3]{36}}\right) > 0 \Rightarrow$ min!

So, $C = \frac{8000}{3\cdot\frac{20}{\sqrt[3]{36}}} + 6\left[\frac{20}{\sqrt[3]{36}}\right]^2$

$\boxed{\approx \$660.39}$

Example: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 1400 - 200p$ units. Each unit costs \$3 to make.

1. What price p should the company charge to maximize their revenue?
2. What price p should the company charge to maximize their profits?

① Revenue = total amount of money the seller brings in from selling their product
= (unit price of product) × (number of units sold)

$R = pq = p(1400-200p) = 1400p - 200p^2$

Domain of R: $[0,\infty)$ (price can't be negative)

Note $R' = 1400 - 400p$

Note $R' \neq$ DNE.

Solve $R' = 0$

↳ $1400 - 400p = 0$

$\Rightarrow p = \frac{1400}{400} = \frac{14}{4} = \frac{7}{2}$ ← verify max

$p = \$\frac{7}{2} = \$3.50$

$R'' = -400$

$R''(7/2) = -400 < 0 \Rightarrow$ max!

② profit = net gain seller makes by selling their product
= revenue − costs

$B = pq - 3q$

$= 1400p - 200p^2 - 3(1400-200p)$

$= 1400p - 200p^2 - 4200 + 600p$

$= -200p^2 + 2000p - 4200$

Domain of B: $[0,\infty)$ (price can't be negative)

Note $B' = -400p + 2000$

Note $B' \neq$ DNE.

Solve $B' = 0$

↳ $0 = -400p + 2000$

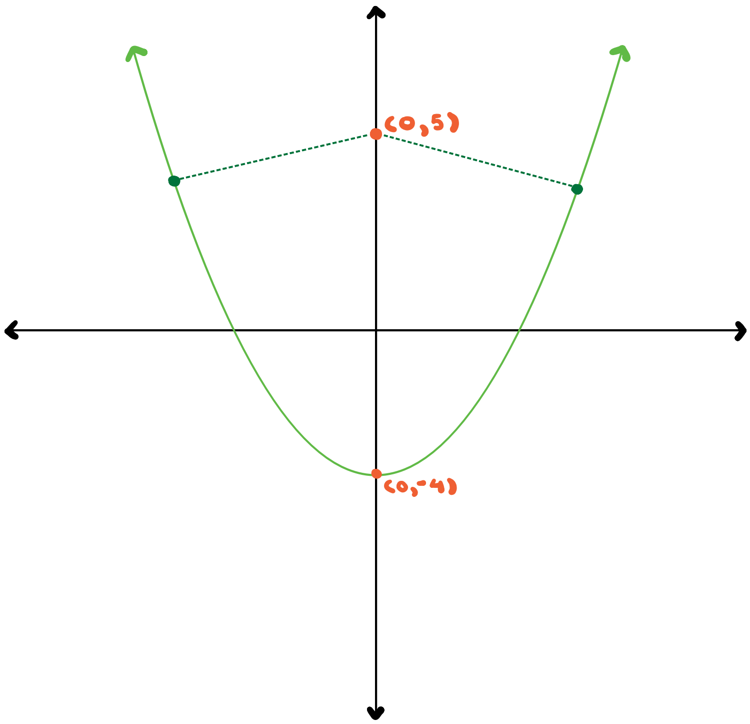
$\Rightarrow p = 5$ ← verify max

$p = \$5$

$B'' = -400$

$B''(5) = -400 < 0 \Rightarrow$ max!

Example: Find the points on the curve $y = x^2 - 4$ closest to the point $(0, 5)$.



Distance between points (x_1, y_1) and (x_2, y_2): $\sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

We want to minimize distance:

$d = \sqrt{(x-0)^2 + (y-5)^2}$ ← obj. function

$y = x^2 - 4$ ← constraint eq.

$$d = \sqrt{x^2 + (\overbrace{x^2 - 4 - 5}^{-9})^2}$$

$$= \sqrt{x^2 + x^4 - 18x^2 + 81}$$

$$= \sqrt{x^4 - 17x^2 + 81}$$

Note: optimizing d is the same as optimizing what's inside the root.

So, use: $D = x^4 - 17x^2 + 81$

No restrictions on x. So, x is in $(-\infty, \infty)$.

$D' = 4x^3 - 34x$

Note $D' \neq$ DNE.

Solve $D' = 0$

↳ $4x^3 - 34x = 0$

$\Rightarrow 2x(2x^2 - 17) = 0$

$\Rightarrow 2x = 0$ or $2x^2 - 17 = 0$

$\Rightarrow x = 0$ $\quad\quad$ $\Rightarrow x^2 = 17/2$

$\Rightarrow x = \pm\sqrt{17/2}$

$$\left(-\sqrt{17/2}, \left(-\sqrt{17/2}\right)^2 - 4\right) = \left(-\sqrt{17/2}, \overbrace{\tfrac{17}{2} - 4}^{9/2}\right)$$

$$\left(\sqrt{17/2}, \left(\sqrt{17/2}\right)^2 - 4\right) = \left(\sqrt{17/2}, 9/2\right)$$

closest points

x	$D(x)$
$-\sqrt{17/2}$	$\frac{289}{4} - 17 \cdot \frac{17}{2} + 81 = \frac{289}{4} - \frac{289}{2} + 81 < 81$
0	81
$\sqrt{17/2}$	$\frac{289}{4} - 17 \cdot \frac{17}{2} + 81 = \frac{289}{4} - \frac{289}{2} + 81 < 81$

mins.