26 Lesson 26

26.1 Antiderivatives and Indefinite Integration

Idea: Antidifferentiation is the reverse process of differentiation, i.e. we want to undo the derivative.

Example: Suppose $f(x) = 6x$. What is a function $F(x)$ such that $F'(x) = f(x)$?

$F(x) = 3x^2$
or
$F(x) = 3x^2 + 1$
or
$F(x) = 3x^2 + 72$

So, $F(x) = 3x^2 + C$ ← any real number

Definition: A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Note: Any two antiderivatives of a function will differ by a constant. In other words, if $F(x)$ is an antiderivative of $f(x)$, then $G(x)$ is an antiderivative of $f(x)$ if and only if $F(x) = G(x) + C$, where C is a constant.

Definition: The process of finding all the antiderivatives of a function is called indefinite integration. We denote this by

"integral of f(x)"

$$\int f(x)dx = F(x) + C,$$

where C is a constant. We define the symbol $\int$ to be an integral sign. The function $f(x)$ is the integrand. The variable x is the integration variable. We call C the constant of integration.

Basic Differentiation Rule	Basic Integration Rule
$\frac{d}{dx}C = 0$	$\int 0\,dx = C$
$\frac{d}{dx}(kx) = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int [kf(x)]\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\ln(x) = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example: Evaluate

$$\int \frac{2x+11}{3}\,dx.$$

$$= \int \left[\frac{2x}{3} + \frac{11}{3}\right] dx$$

$$= \int \frac{2x}{3}\,dx + \int \frac{11}{3}\,dx$$

$$= \frac{2}{3}\int x\,dx + \frac{11}{3}\int 1\,dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2} + c_1\right] + \frac{11}{3}\left[x + c_2\right]$$

$$= \frac{2x^2}{6} + \frac{2}{3}c_1 + \frac{11}{3}x + \frac{11}{3}c_2$$

$$= \frac{x^2}{3} + \frac{11}{3}x + c \qquad \text{let } c = \frac{2}{3}c_1 + \frac{11}{3}c_2$$

Example: Evaluate

$$\int \left(8x^7 - \frac{\sqrt[3]{x^2}}{5}\right)\,dx.$$

$$= \int 8x^7\,dx - \int \frac{x^{2/3}}{5}\,dx$$

$$= 8\int x^7\,dx - \frac{1}{5}\int x^{2/3}\,dx$$

$$= 8\left[\frac{x^8}{8} + c_1\right] - \frac{1}{5}\left[\underbrace{\frac{1}{(5/3)}}_{=\,3/5} x^{\overbrace{2/3+3/3}^{5/3}} + c_2\right]$$

$$= x^8 + 8c_1 - \frac{3}{25}x^{5/3} - \frac{1}{5}c_2$$

$$= x^8 - \frac{3}{25}\sqrt[3]{x^5} + C$$

Example: Evaluate

$$\int \frac{\cos(x)\tan(x) - 3\sin(x)}{2}\,dx.$$

$$= \frac{1}{2}\int \left[\cos(x)\overbrace{\tan(x)}^{=\,\frac{\sin(x)}{\cos(x)}} - 3\sin(x)\right]dx$$

$$= \frac{1}{2}\int \left[\overbrace{\sin(x) - 3\sin(x)}^{=\,-2\sin(x)}\right]dx$$

$$= \frac{1}{2}\cdot(-2)\int \sin(x)\,dx$$

$$= \frac{1}{2}\cdot(-2)\cdot\left[-\cos(x) + c\right]$$

$$= \cos(x) + C$$

Example: Evaluate

$$\int \frac{x^4 - x^5}{\sqrt{x}} dx.$$

$= \frac{x^4}{x^{1/2}} - \frac{x^5}{x^{1/2}} = x^{7/2} - x^{9/2}$

$= \int (x^{7/2} - x^{9/2}) dx$

$= \int x^{7/2} dx - \int x^{9/2} dx$

$= \frac{1}{(9/2)} x^{9/2} - \frac{1}{(11/2)} x^{11/2} + C$

$= \frac{2}{9}\sqrt{x^9} - \frac{2}{11}\sqrt{x^{11}} + C$

Example: Evaluate

$$\int \frac{8 + 3xe^x}{x} dx.$$

$= \int \left[\frac{8}{x} + \frac{3xe^x}{x} \right] dx$ ($\frac{3xe^x}{x} = 3e^x$)

$= \int \frac{8}{x} dx + \int 3e^x dx$

$= 8\int \frac{1}{x} dx + 3\int e^x dx$

$= 8\ln|x| + 3e^x + C$