27 Lesson 27

27.1 Antiderivatives and Indefinite Integration

Goal: Instead of solving for a family of antiderivatives (i.e. a general solution), we will solve for a specific antiderivative using an initial condition. We call this sort of problem an initial value problem.

Example: Suppose $f'(x) = 3x^2 + \frac{4}{x}$ and $f(1) = -1$. Find $f(2)$.

$f(x) = \int f'(x)\,dx$

$= \int (3x^2 + \frac{4}{x})\,dx$

$= 3\int x^2\,dx + 4\int \frac{1}{x}\,dx$

$= 3\cdot\frac{x^3}{3} + 4\ln|x| + C$

$= x^3 + 4\ln|x| + C$

$-1 = f(1)$

$= 1^3 + 4\ln|1| + C$

$= 1^3 + 4\cdot 0 + C$

$\Rightarrow C = -1-1 = -2$

So, $f(x) = x^3 + 4\ln|x| - 2$

$\Rightarrow f(2) = 2^3 + 4\ln|2| - 2$

$= 8 + 4\ln(2) - 2$

$= \boxed{6 + 4\ln(2)}$

Example: Suppose $y'' = 2\sin(x) + 3\pi x^2$ where $y'(0) = 1$ and $y(\pi) = -1$. Find $y(0)$.

$y' = \int y''\,dx$

$= \int (2\sin x + 3\pi x^2)\,dx$

$= 2\int \sin x\,dx + 3\pi\int x^2\,dx$

$= -2\cos x + 3\pi\frac{x^3}{3} + C_1$

$= -2\cos x + \pi x^3 + C_1$

$1 = y'(0)$

$= -2\cos(0) + \pi\cdot 0^3 + C_1$

$= -2 + C_1$

$\Rightarrow C_1 = 3$

So, $y' = -2\cos x + \pi x^3 + 3$

$y = \int y'\,dx$

$= \int (-2\cos x + \pi x^3 + 3)\,dx$

$= -2\int \cos x\,dx + \pi\int x^3\,dx + \int 3\,dx$

$= -2\sin x + \pi\frac{x^4}{4} + 3x + C_2$

$-1 = y(\pi)$

$= -2\sin\pi + \pi\cdot\frac{\pi^4}{4} + 3\pi + C_2$

$= 0 + \frac{\pi^5}{4} + 3\pi + C_2$

$\Rightarrow C_2 = -1 - \frac{\pi^5}{4} - 3\pi$

$\Rightarrow y = -2\sin x + \pi\frac{x^4}{4} + 3x - 1 - \frac{\pi^5}{4} - 3\pi$

$\Rightarrow y(0) = -2\cdot 0 + \pi\cdot\frac{0^4}{4} + 3\cdot 0 - 1 - \frac{\pi^5}{4} - 3\pi$

$= \boxed{-1 - \frac{\pi^5}{4} - 3\pi}$

Example: The rate of growth $\frac{dP}{dt}$ of a population of bacteria is proportional to the square root of t with a constant coefficient of 3, where P is the population size and t is the time in days. If the initial size of the population is 1200, what is the population after 4 days?

$\hookrightarrow P(0) = 1200$

$$\frac{dP}{dt} = 3\sqrt{t}$$

$$\begin{aligned} P(t) &= \int P'(t)\,dt \\ &= \int 3\sqrt{t}\,dt \\ &= 3\int t^{1/2}\,dt \\ &= 3\cdot\frac{t^{3/2}}{3/2} + C \\ &= 3\cdot\frac{2}{3}t^{3/2} + C \\ &= 2t^{3/2} + C \end{aligned}$$

So, $1200 = P(0)$
$= 2\cdot 0^{3/2} + C$

$\Rightarrow C = 1200$

$\Rightarrow P(t) = 2t^{3/2} + 1200$

$$\begin{aligned} \Rightarrow P(4) &= 2\sqrt{4^3} + 1200 \\ &= 2\sqrt{64} + 1200 \\ &= 16 + 1200 \\ &= 1216 \end{aligned}$$

Example: A hot air balloon is rising vertically with a velocity of 4 ft/sec. A small ball is released from the hot air balloon at the instant when it is 462 feet above the ground. Use $a(t) = -32$ ft/sec^2 as the acceleration due to gravity.

$\hookrightarrow v(0) = 4$

$\hookrightarrow p(0) = 462$

1. How long will it take the ball to reach the ground? 5.5 seconds
2. At what velocity will it hit the ground? −172 ft/sec

$a(t) = v'(t)$
$v(t) = p'(t)$

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int -32\,dt \\ &= -32t + c_1 \end{aligned}$$

$$\begin{aligned} 4 &= v(0) \\ &= -32\cdot 0 + c_1 \\ &= c_1 \end{aligned}$$

$\Rightarrow v(t) = -32t + 4$

$$\begin{aligned} p(t) &= \int v(t)\,dt \\ &= \int (-32t + 4)\,dt \\ &= -16t^2 + 4t + c_2 \end{aligned}$$

$$\begin{aligned} 462 &= p(0) \\ &= -16\cdot 0^2 + 4\cdot 0 + c_2 \end{aligned}$$

$\Rightarrow 462 = c_2$

$\Rightarrow p(t) = -16t^2 + 4t + 462$

① When is $p(t) = 0$?

$$\begin{aligned} 0 &= -16t^2 + 4t + 462 \\ &= -2(8t^2 - 2t - 231) \end{aligned}$$

$\Rightarrow 0 = 8t^2 - 2t - 231$

$$\begin{aligned} \Rightarrow t &= \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 8\cdot(-231)}}{2\cdot 8} \\ &= \frac{2 \pm \sqrt{7396}}{16} \\ &= \frac{1}{8} \pm \frac{86}{16} \\ &= \frac{1}{8} \pm \frac{43}{8} \end{aligned}$$

$= \frac{1 \pm 43}{8}$ only want positive

So $t = \cancel{\frac{-42}{8}}, \frac{44}{8}$ 5.5

② want $v(5.5)$

$v(5.5) = -32 \cdot 5.5 + 4$

$= -172$ ft/sec