28 Lesson 28

28.1 Area and Riemann Sums

Notation: $\sum$, namely the sigma symbol, means to "sum up."

Example: Evaluate

$$\sum_{i=1}^{5}(2i+1).$$

← plug in $i=1,\ldots,5$ into $2i+1$ and add all the terms

$$= \underbrace{(2\cdot 1+1)}_{2+1=3} + \underbrace{(2\cdot 2+1)}_{4+1=5} + \underbrace{(2\cdot 3+1)}_{6+1=7} + \underbrace{(2\cdot 4+1)}_{8+1=9} + \underbrace{(2\cdot 5+1)}_{10+1=11}$$

$$= 3+5+7+9+11$$

$$= \boxed{35}$$

Example: Using sigma notation, rewrite the following sum.

$$\sqrt{\cos(x)+1} + \sqrt{\cos(2x)+1} + \sqrt{\cos(3x)+1} + \cdots + \sqrt{\cos(nx)+1}$$

Look at each term in sum. What's changing?

↳ $\cos(ix)$ where $i=1,\ldots,n$

$$= \boxed{\sum_{i=1}^{n}\sqrt{\cos(ix)+1}}$$

Goal: Approximate signed area under the curve of a function. This is the area enclosed by the function and the x-axis with a sign.

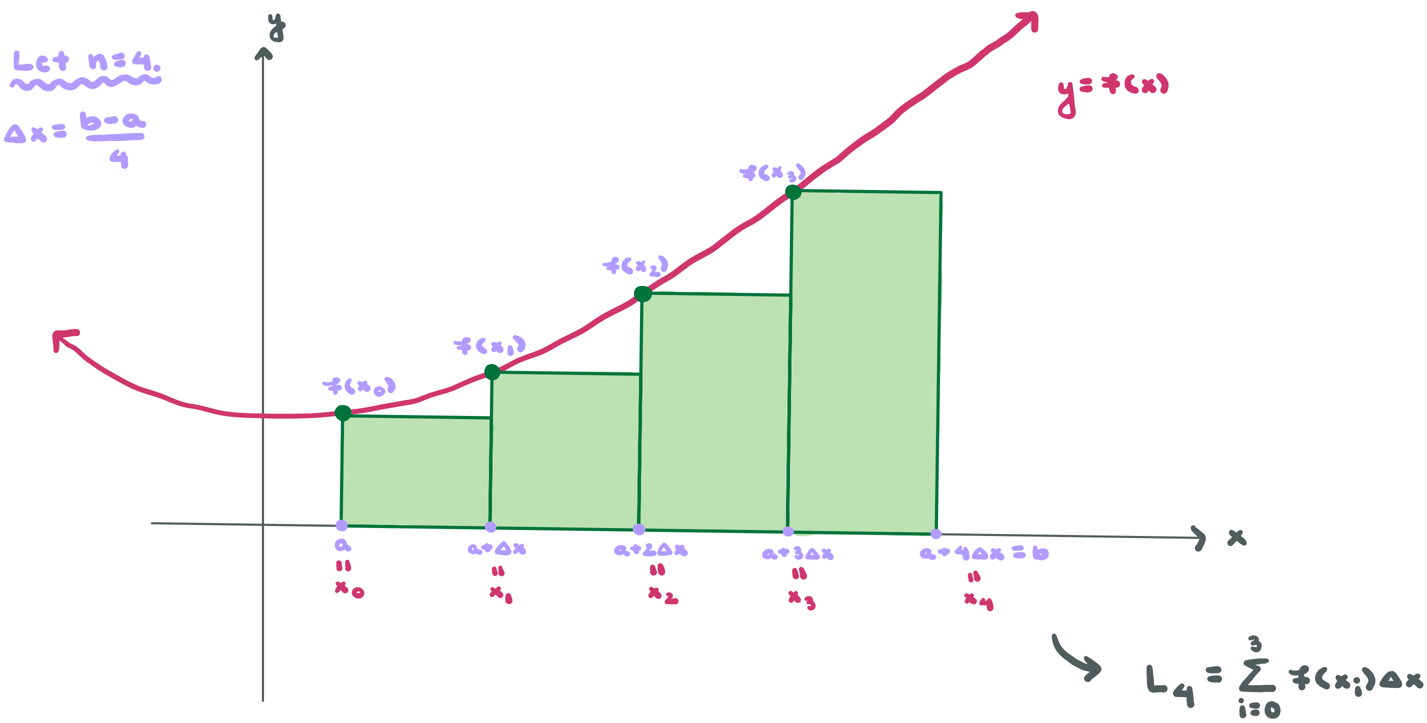
- If the function is above the x-axis, then the area is positive.
- If the function is below the x-axis, then the area is negative.

Idea: We will approximate the (signed) area between a curve and the x-axis using rectangles.

Strategies: Suppose $f(x)$ is a function on the interval $[a, b]$. We want to approximate the (signed) area between $f(x)$ and the x-axis on $[a, b]$ using n rectangles. We will use *left* and *right* Riemann sums.

Divide the interval $[a, b]$ into n sub-intervals of width $\Delta x = \frac{b-a}{n}$.

Left Riemann Sum: Denote the left Riemann sum by L_n. Construct rectangles of width Δx from the left endpoint of each sub-interval. The height of each rectangle is given by the value of $f(x)$, where x is the left endpoint of each sub-interval.



It follows that

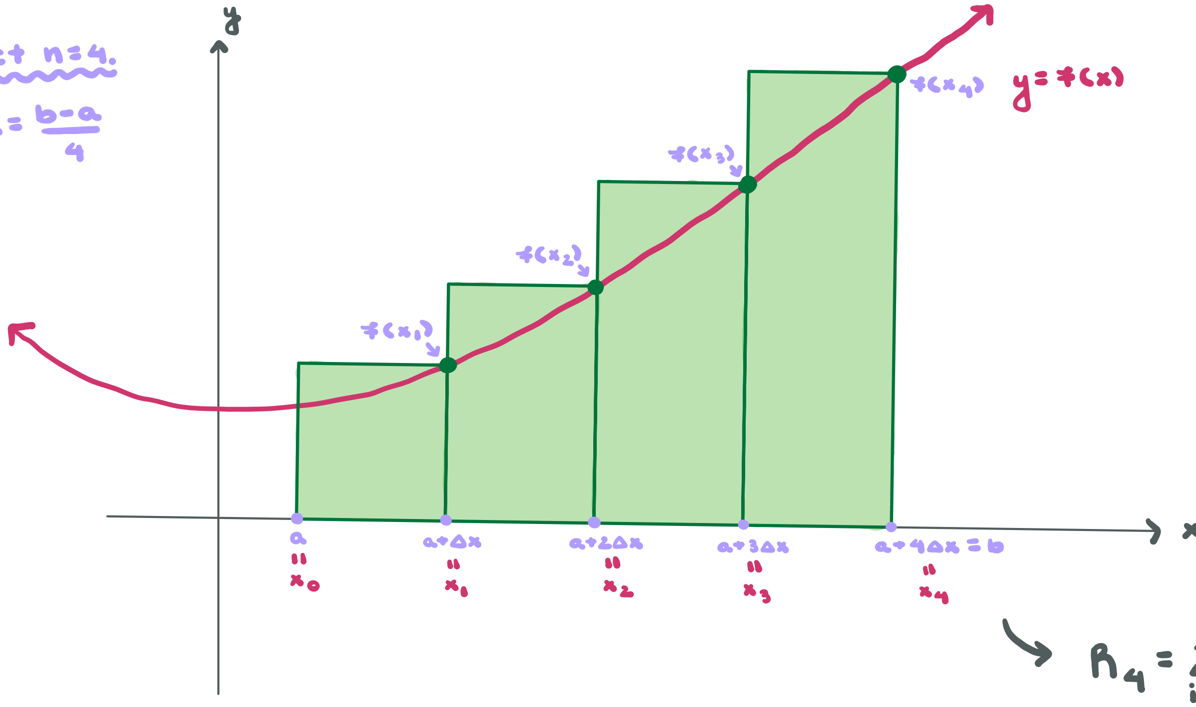
$$L_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x,$$

where $x_i = a + i \cdot \Delta x$.

Right Riemann Sum: Denote the right Riemann sum by R_n. Construct rectangles of width Δx from the right endpoint of each sub-interval. The height of each rectangle is given by the value of $f(x)$, where x is the right endpoint of each sub-interval.

Let $n=4$.

$\Delta x = \frac{b-a}{4}$



$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

It follows that

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x,$$

where $x_i = a + i \cdot \Delta x$.

Example: Use the left and right Riemann sums with 3 rectangles to estimate the (signed) area under the curve of $y = x + 2$ on the interval $[1, 7]$.

$n = 3$

$[a,b] = [1,7]$

$\Delta x = \frac{b-a}{n} = \frac{7-1}{3} = 2$

$x_i = a + i\Delta x = 1 + i2$

$L_3 = \sum_{i=0}^{2} y(x_i)\underbrace{\Delta x}_{2}$

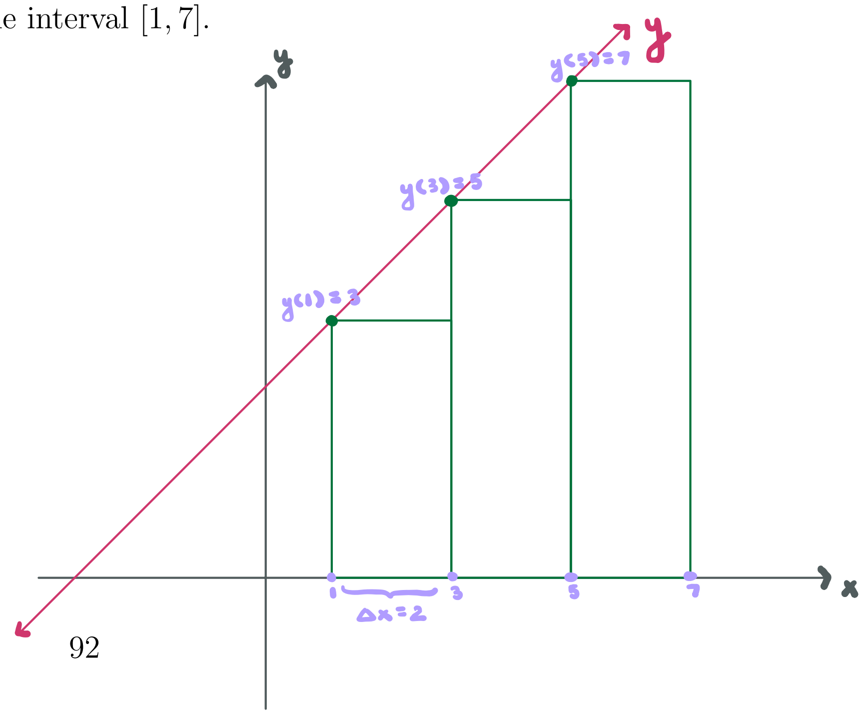
$= \sum_{i=0}^{2} [1+i2+2]\cdot 2$

$= 2\sum_{i=0}^{2} (3+i2)$

$= 2\left[\underbrace{(3+0\cdot 2)}_{3} + \underbrace{(3+1\cdot 2)}_{5} + \underbrace{(3+2\cdot 2)}_{7}\right]$

$= 2 \cdot 15$

$= \boxed{30}$



$$R_3 = \sum_{i=1}^{3} y(x_i)\underbrace{\Delta x}_{2}$$

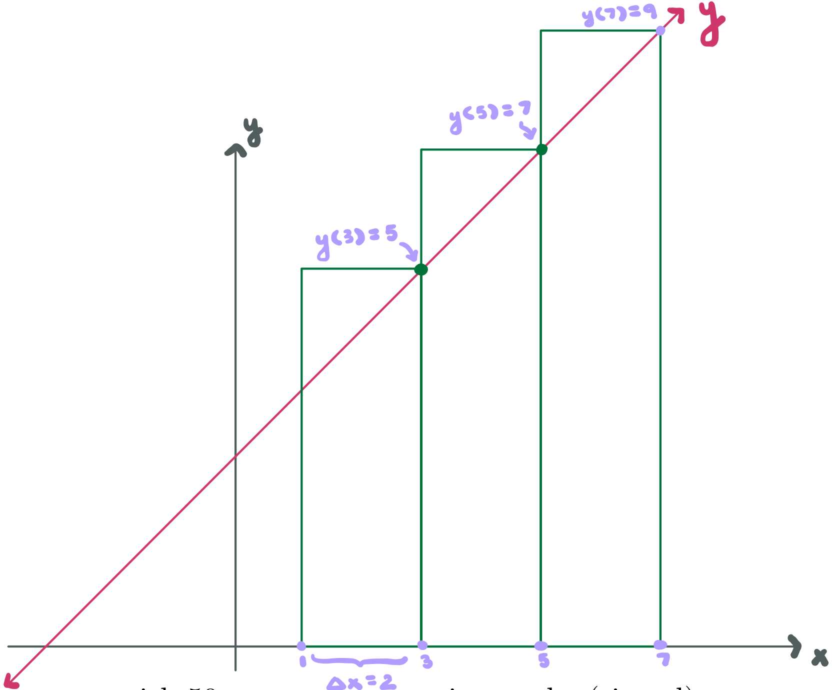
$$= \sum_{i=1}^{3} [1+i2+2]\cdot 2$$

$$= 2\sum_{i=1}^{3} (3+i2)$$

$$= 2\Big[\underbrace{(3+1\cdot 2)}_{5} + \underbrace{(3+2\cdot 2)}_{7} + \underbrace{(3+3\cdot 2)}_{9}\Big]$$

$$= 2\cdot 21$$

$$= \boxed{42}$$



Example: Use the left and right Riemann sums with 50 rectangles to estimate the (signed) area under the curve of $y = 3x^2 + x - 1$ on the interval $[0, 150]$.

$n = 50$

$[a, b] = [0, 150]$

$$\Delta x = \frac{b-a}{n} = \frac{150-0}{50} = 3$$

$$x_i = a + i\Delta x = 0 + i3 = i3$$

$$L_{50} = \sum_{i=0}^{49} y(\overbrace{x_i}^{i3})\Delta x$$

$$= \sum_{i=0}^{49} (3\overbrace{(i3)^2}^{=9i^2} + i3 - 1)\cdot 3$$

$$= \boxed{3\sum_{i=0}^{49} (27i^2 + i3 - 1)}$$

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