2 Lesson 2

2.1 Limits

Definition: The limit of a function $f(x)$ is the value that $f(x)$ approaches as x approaches a particular value. If $f(x)$ approaches L as x approaches c, we say that the limit of $f(x)$ as x approaches c is L, denoted by

$$\lim_{x\to c} f(x) = L.$$

Note: $f(x)$ does not need to be defined at $x = c$ for the limit to exist.

Definition: If $f(x)$ increases or decreases without bound as x approaches c, then $\lim_{x\to c} f(x)$ is an infinite limit. If $f(x)$ increases without bound, $\lim_{x\to c} f(x) = \infty$. If $f(x)$ decreases without bound, $\lim_{x\to c} f(x) = -\infty$.

Note: A limit does not need to exist, i.e. a limit can evaluate to DNE (does not exist). This happens when $f(x)$ does not approach a ~~concrete~~ specific value as x approaches c.

2.2 Finding Limits Numerically

Idea: To estimate $\lim_{x\to c} f(x)$, we can evaluate $f(x)$ at values of x which get closer and closer to c.

Example 1: Evaluate the following limit numerically:

$$\lim_{x\to 1}(x+3) = 4$$

x	.9	.99	.999	.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	3.9	3.99	3.999	3.9999	-	4.0001	4.001	4.01	4.1

Notice that $1 + 3 = 4$.

Example 2: Evaluate the following limit numerically:

$$\lim_{x\to 2} \frac{x^2 - 2x}{x - 2} = 2$$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	1.9999	-	2.0001	2.001	2.01	2.1

Notice that $\frac{x^2-2x}{x-2}$ is not defined at $x = 2$.

Example 3: Evaluate the following limit numerically:

$$\lim_{x\to -4} \frac{8}{(x+4)^2} = \infty$$

x	-4.1	-4.01	-4.001	-4.0001	-4	-3.9999	-3.999	-3.99	-3.9
$f(x)$	800	80,000	$\underbrace{8{,}000{,}000}_{=8\times10^6}$	8×10^8	-	8×10^8	8×10^6	80,000	800

Notice that $\frac{8}{(x+4)^2}$ is not defined at $x = -4$.

2.3 One-sided Limits

Definition: A one-sided limit is the value a function $f(x)$ approaches as x approaches a particular value from the left or right.

Definition (left-sided limit): If $f(x)$ approaches L as x approaches c from the left, we write

$$\lim_{x\to c^-} f(x) = L.$$

Definition (right-sided limit): If $f(x)$ approaches L as x approaches c from the right, we write

$$\lim_{x\to c^+} f(x) = L.$$

Example 4: Suppose

$$f(x) = \begin{cases} 5 & \text{if } x > 1, \\ x & \text{if } x \leq 1. \end{cases}$$

Evaluate the following limits numerically:

$\lim_{x\to 1^-} f(x) =$ 1 $\qquad \lim_{x\to 1^+} f(x) =$ 5 $\qquad \lim_{x\to 1} f(x) =$ DNE

x	.9	.99	.999	.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	.9	.99	.999	.9999	-	5	5	5	5

Note:

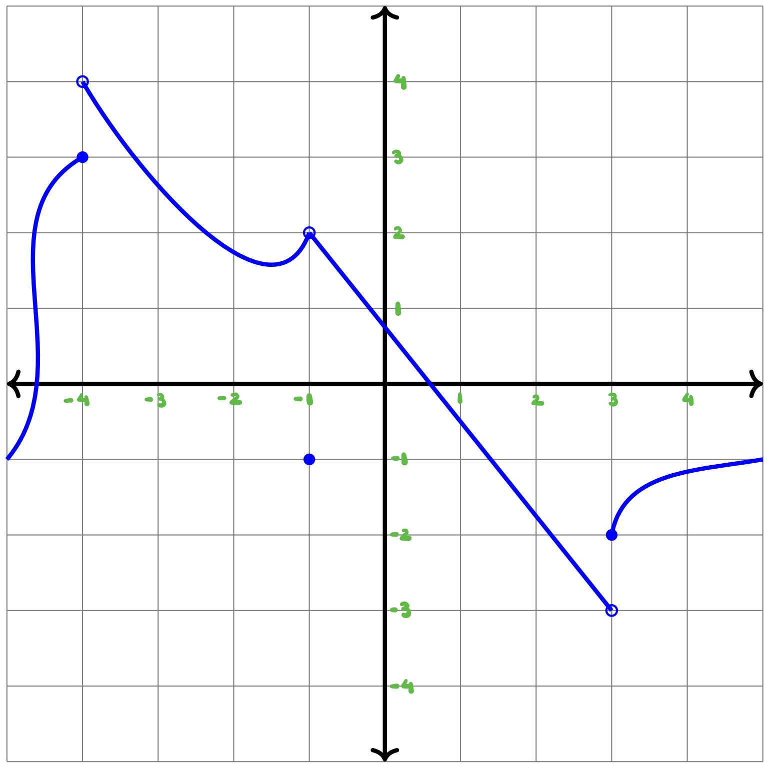
$$\lim_{x\to c} f(x) = L \quad \text{if and only if} \quad \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$$

Note that we allow $L = \pm\infty$.

2.4 Finding Limits Graphically

Idea: We look at the portion of the curve of $f(x)$ near $x = c$ and observe what the function value approaches as x gets closer to c, from both the left and right.

Example 5:



Find the following limits using the graph of $f(x)$ above:

$\lim\limits_{x \to -4^-} f(x) =$ 3

$\lim\limits_{x \to -4^+} f(x) =$ 4

$\lim\limits_{x \to -4} f(x) =$ DNE

$f(-4) =$ 3

$\lim\limits_{x \to -1^-} f(x) =$ 2

$\lim\limits_{x \to -1^+} f(x) =$ 2

$\lim\limits_{x \to -1} f(x) =$ 2

$f(-1) =$ -1

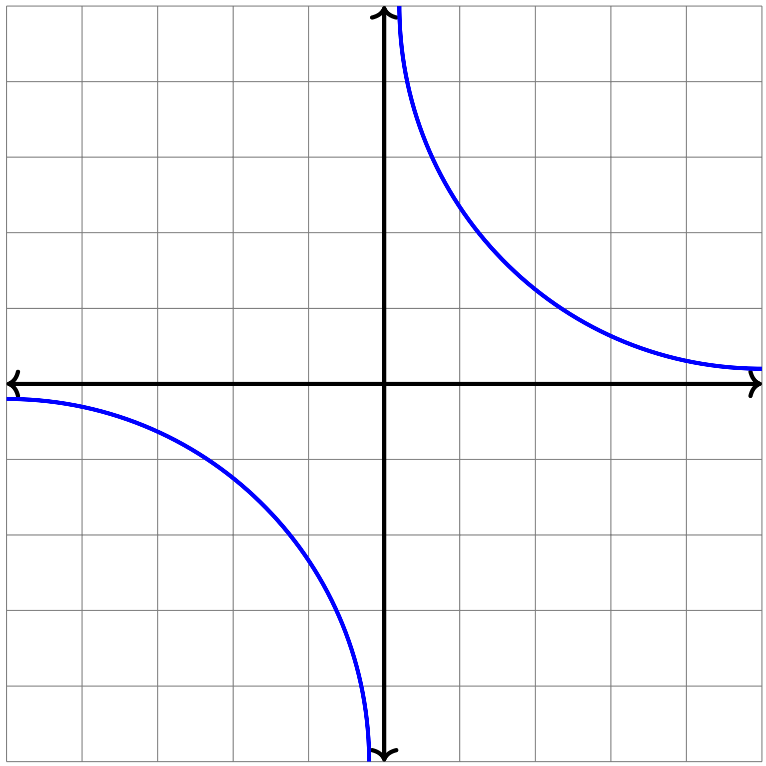
$\lim\limits_{x \to 3^-} f(x) =$ -3

$\lim\limits_{x \to 3^+} f(x) =$ -2

$\lim\limits_{x \to 3} f(x) =$ DNE

$f(3) =$ -2

Example 6:



Find the following limits using the graph of $f(x)$ above:

$\lim_{x\to 0^-} f(x) =$ $-\infty$

$\lim_{x\to 0^+} f(x) =$ ∞

$\lim_{x\to 0} f(x) =$ DNE

$f(0) =$ undefined