31 Lesson 31

31.1 Fundamental Theorem of Calculus

Natural Question: Are antiderivatives somehow connected to definite integrals?
Answer: Yes!

Theorem: Suppose $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note: What do we mean by *an antiderivative*? We can choose any value for the constant of integration C. Suppose that $F(x) + C$ is another antiderivative of $f(x)$, where C is any constant. Observe that

$$\int_a^b f(x)\,dx = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$

For this reason, we tend to let $C = 0$.

Notation:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example: Evaluate

$$\int_2^4 (x^3 + 2x)\,dx.$$

$$= \int_2^4 x^3\,dx + \int_2^4 2x\,dx$$

$$= \frac{x^4}{4}\Big|_2^4 + x^2\Big|_2^4$$

$$= \left(\frac{4^4}{4} - \frac{2^4}{4}\right) + (4^2 - 2^2)$$

$$= 4^3 - 4 + 4^2 - 4$$

$$= 4^3 - 8 + 16$$

$$= 64 + 8$$

$$= \boxed{72}$$

Example: Evaluate

$$\int_0^{\overset{\pi/4}{\cancel{\frac{\pi}{2}}}} (5\cos(x) - \sec^2(x))\, dx.$$

$$= \left(5\sin(x) - \tan(x)\right)\Big|_0^{\pi/4}$$

$$= \Big(\underbrace{5\sin\left(\tfrac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}} - \underbrace{\tan\left(\tfrac{\pi}{4}\right)}_{\frac{\sqrt{2}}{\sqrt{2}} = 1}\Big) - \Big(5\underbrace{\sin(0)}_{0} - \underbrace{\tan(0)}_{0}\Big)$$

$$= \boxed{\frac{5\sqrt{2}}{2} - 1}$$

Example: Evaluate

$$\int_3^6 (8e^x - x + 1)\, dx.$$

$$= \left(8e^x - \frac{x^2}{2} + x\right)\Big|_3^6$$

$$= \left(8e^6 - \frac{6^2}{2} + 6\right) - \left(8e^3 - \frac{3^2}{2} + 3\right)$$

$$= 8e^6 - 18 + 6 - 8e^3 + \frac{9}{2} - 3$$

$$= 8e^3(e^3 - 1) \underbrace{- 15 + \frac{9}{2}}_{-\frac{30}{2} + \frac{9}{2} = \frac{-21}{2}}$$

$$= \boxed{8e^3(e^3 - 1) - \frac{21}{2}}$$

Example: Evaluate

$$\frac{x^4}{x^2} + \frac{x^{1/2}}{x^2} = x^2 + x^{-3/2}$$

$$\int_1^2 \frac{x^4+\sqrt{x}}{x^2}\,dx.$$

$$= \int_1^2 (x^2 + x^{-3/2})\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^{-1/2}}{(-1/2)}\right)\Big|_1^2$$

$$= \left(\frac{x^3}{3} - 2x^{-1/2}\right)\Big|_1^2$$

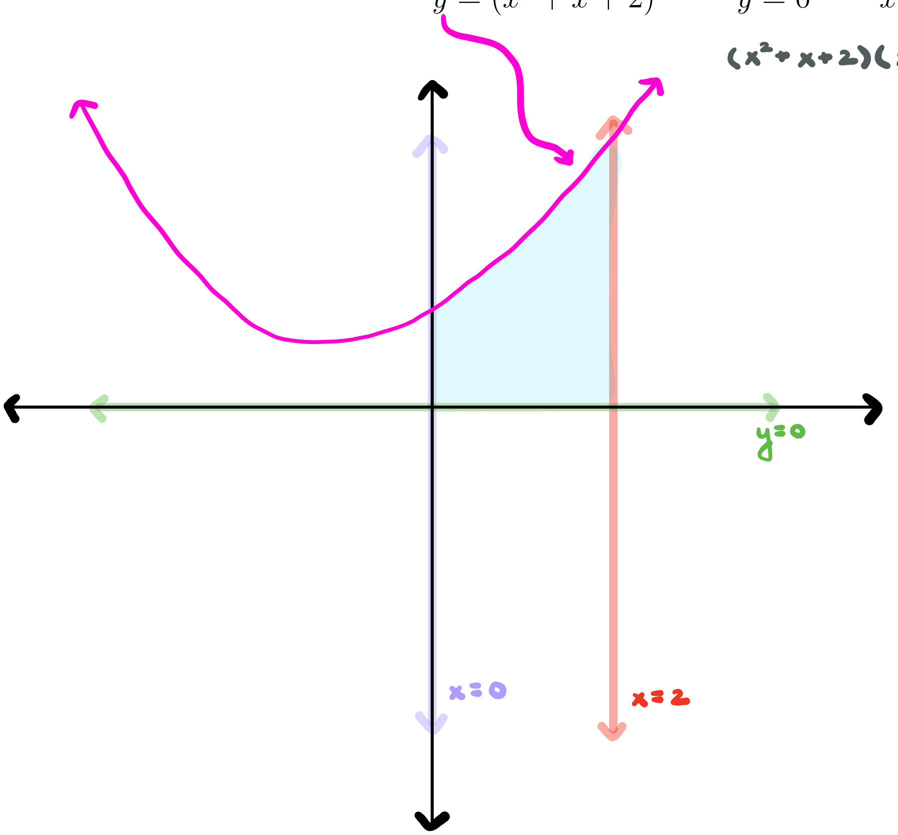
$$= \left(\frac{2^3}{3} - \frac{2}{\sqrt{2}}\right) - \left(\underbrace{\frac{1^3}{3} - \frac{2}{\sqrt{1}}}_{\frac{1}{3} - 2 = -\frac{5}{3}}\right)$$

$$= \frac{8}{3} - \frac{2}{\sqrt{2}} + \frac{5}{3}$$

$$= \boxed{\frac{13}{3} - \sqrt{2}}$$

Example: Find the area of the region bounded by the graphs of the following equations.

$$y = (x^2+x+2)^2 \qquad y = 0 \qquad x = 0 \qquad x = 2$$



$$(x^2+x+2)(x^2+x+2) = x^4 + x^3 + 2x^2$$
$$+ x^3 + x^2 + 2x$$
$$+ 2x^2 + 2x + 4$$
$$= x^4 + 2x^3 + 5x^2 + 4x + 4$$

$$\int_0^2 (x^4 + 2x^3 + 5x^2 + 4x + 4)\,dx$$

$$= \frac{x^5}{5} + \frac{x^4}{2} + \frac{5}{3}x^3 + 2x^2 + 4x \Big|_0^2$$

$$= \frac{2^5}{5} + \frac{2^4}{2} + \frac{5}{3}\cdot 2^3 + \underbrace{2\cdot 2^2}_{8} + \underbrace{4\cdot 2}_{8}$$

$$= \frac{32}{5} + \underbrace{2^3}_{8} + \frac{40}{3} + 16$$

$$= 24 + \frac{96}{15} + \frac{200}{15}$$

$$= 24 + \frac{296}{15}$$

$$= \frac{360}{15} + \frac{296}{15} = \boxed{\frac{656}{15}}$$