32 Lesson 32

32.1 Fundamental Theorem of Calculus

Example: The growth rate $P'(t)$ of the population of a town is given by

$$P'(t) = 100(t+3),$$

where t is time in years. How does $\underbrace{\text{the population}}_{\text{given by } P(t)}$ change from $t = 1$ year to $t = 5$ years?

$$P(5) - P(1) = \int_1^5 P'(t)\,dt$$

$$= \int_1^5 \overbrace{100(t+3)}^{100t+300}\,dt$$

$$= 50t^2 + 300t \Big|_1^5$$

$$= (50 \cdot \underbrace{5^2}_{25} + 300 \cdot 5) - (50 \cdot 1^2 + 300 \cdot 1)$$

$$= 1250 + 1500 - 50 - 300$$

$$= 2750 - 350$$

$$= \boxed{2400}$$

Example: The velocity function $v(t)$, in meters per second, of a moving train is given by

$$v(t) = 4t^2 - 3.$$

1. Find the displacement of the ~~particle~~ train $\underbrace{\text{from time } t = 1 \text{ to time } t = 5}_{\text{Find } p(5)-p(1)}$.
2. Find the time when the displacement is zero after the ~~particle~~ train starts moving.

① $$p(5) - p(1) = \int_1^5 (4t^2 - 3)\,dt$$

$$= \frac{4}{3}t^3 - 3t \Big|_1^5$$

$$= \left[\frac{4}{3} \cdot 5^3 - 3 \cdot 5\right] - \left[\frac{4}{3} - 3\right]$$

$$= \frac{\overbrace{4 \cdot 125}^{500}}{3} - 15 - \frac{4}{3} + 3$$

$$= \frac{496}{3} - \underbrace{12}_{\frac{36}{3}}$$

$$= \boxed{\frac{460}{3} \text{ meters}}$$

② $$0 = p(b) - p(0) = \int_0^b (4t^2 - 3)\,dt$$

$$= \frac{4}{3}t^3 - 3t \Big|_0^b$$

$$= \frac{4}{3}b^3 - 3b$$

$$= b\left(\frac{4}{3}b^2 - 3\right)$$

$$\Rightarrow \quad b = 0 \quad \text{or} \quad \underbrace{\frac{4}{3}b^2 - 3 = 0}$$

$$\Rightarrow b^2 = \frac{9}{4}$$

$$\Rightarrow b = \pm 3/2$$

Pick positive!

$$\boxed{b = \underbrace{3/2}_{1.5} \text{ seconds}}$$

Example: The acceleration $a(t)$ of a bicycle after a bicyclist steps on the brakes is given by

$$a(t) = -(t-8)^2,$$

where t is time in seconds. If distance is measured in feet, what is the decrease in velocity 3 seconds after the brake is applied?

129 ft/sec.

$$v(3)-v(0) = \int_0^3 \overbrace{-(t-8)^2}\,dt$$

$-(t-8)(t-8) = -(t^2-16t+64)$

$= -t^2+16t-64$

$$= -\frac{t^3}{3} + 8t^2 - 64t \Big|_0^3$$

$$= -\frac{3^3}{3} + 8\cdot 3^2 - 64\cdot 3$$

$$= -9 + 72 - 192$$

$$= -9 - 120$$

$$= \boxed{-129 \text{ ft/sec.}}$$

Example: A hose is turned on at 10:00am and water starts to flow into a pool at the rate of $r(t) = 4\sqrt{t}$, where t is time in hours after 10:00am. Suppose that $r(t)$ is measured using ft^3/hr.

1. How much water (in cubic feet) flows into the pool between 10:00am ($t=0$) and 2:00pm ($t=4$)?
2. How many hours after 10:00am will there be 100 cubic feet of water in the pool?

Let $p(t)$ be an antiderivative of $r(t)$.

① $$p(4)-p(0) = \int_0^4 \overbrace{4\sqrt{t}}^{4t^{1/2}}\,dt$$

$$= \frac{4t^{3/2}}{3/2}\Big|_0^4$$

$$= \frac{8}{3}t^{3/2}\Big|_0^4$$

$$= \frac{8}{3}\cdot 4^{3/2}$$

$$= \frac{8}{3}\cdot 2^3$$

$$= \boxed{\frac{64}{3} \text{ ft}^3}$$

② $$100 = p(b)-p(0)$$

$$= \int_0^b 4\sqrt{t}\,dt$$

$$= \frac{8}{3}t^{3/2}\Big|_0^b$$

$$= \frac{8}{3}b^{3/2}$$

$$\Rightarrow 100 = \frac{8}{3}b^{3/2}$$

$$\Rightarrow \frac{300}{8} = b^{3/2} \qquad \frac{75}{2}$$

$$\Rightarrow \boxed{b = \left(\frac{75}{2}\right)^{2/3} \text{ hours}} \approx 11.2$$