33 Lesson 33

33.1 Numerical Integration

33.1.1 Trapezoidal Rule

Goal: Approximate definite integrals using trapezoids.

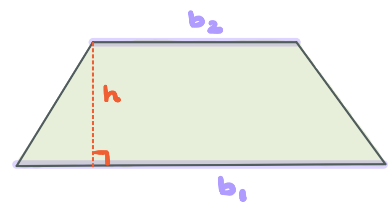
Suppose $f(x)$ is continuous on $[a, b]$, and we want to compute

$$\int_a^b f(x)\,dx.$$

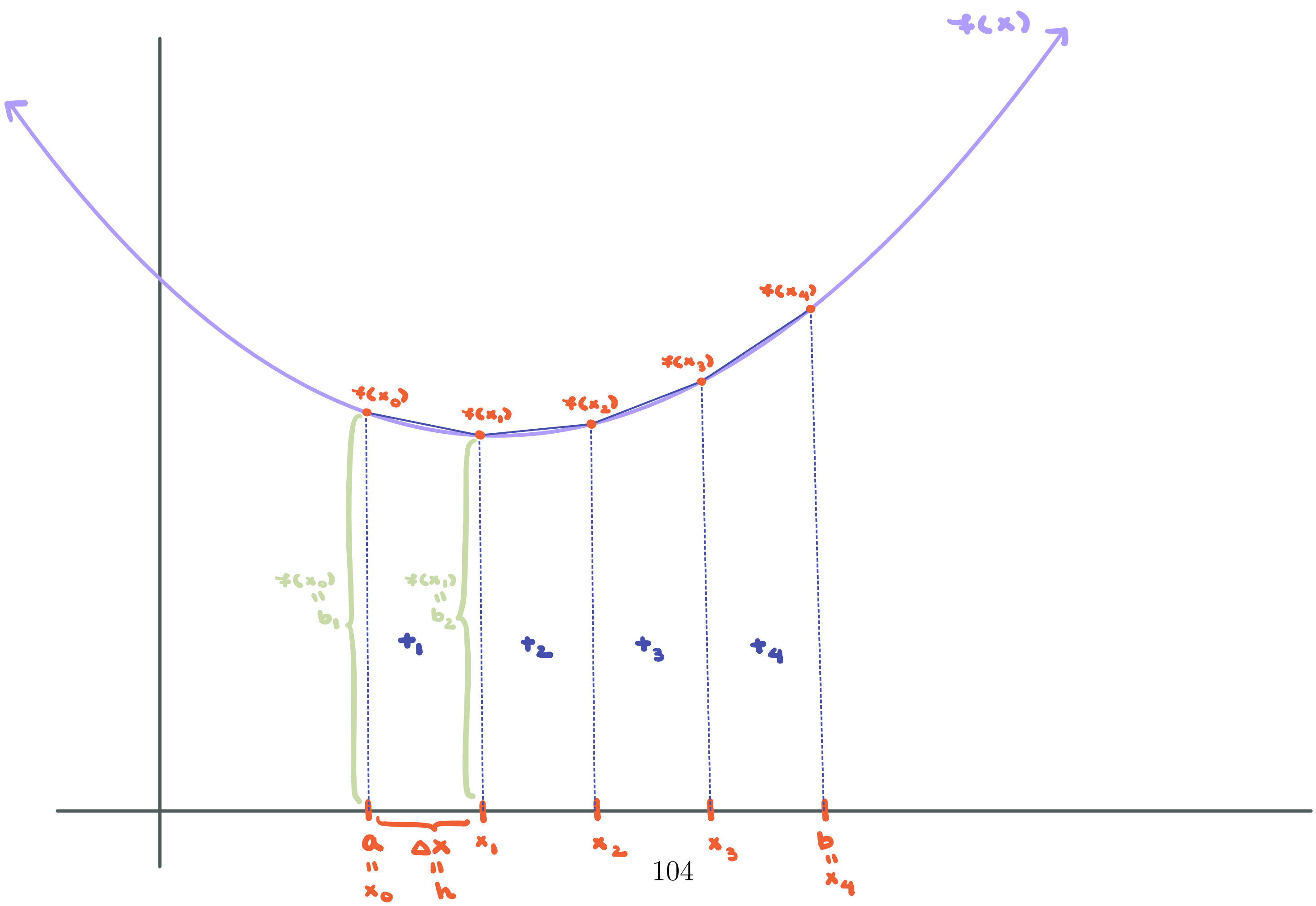
We divide the interval $[a, b]$ into n sub-intervals, each of length $\Delta x = \frac{b-a}{n}$. We construct a trapezoid on each sub-interval and compute its area.

Note: The area of a trapezoid is given by

$$A = \frac{1}{2}(\text{height})(\text{base 1} + \text{base 2}).$$



Let's do an example where $n = 4$.



Let t_i denote the area of each trapezoid. So, we have that

$$t_1 = \frac{1}{2}(f(x_0) + f(x_1))\Delta x,$$
$$t_2 = \frac{1}{2}(f(x_1) + f(x_2))\Delta x,$$
$$t_3 = \frac{1}{2}(f(x_2) + f(x_3))\Delta x,$$
$$t_4 = \frac{1}{2}(f(x_3) + f(x_4))\Delta x,$$

where $x_i = a + i \cdot \Delta x$. The estimate of the definite integral using the trapezoidal rule with $n = 4$ is given by

$$T_4 = \frac{1}{2}(f(x_0) + f(x_1))\Delta x + \frac{1}{2}(f(x_1) + f(x_2))\Delta x + \frac{1}{2}(f(x_2) + f(x_3))\Delta x + \frac{1}{2}(f(x_3) + f(x_4))\Delta x$$
$$= \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)).$$

General Formula: Suppose $f(x)$ is continuous on $[a, b]$. The trapezoidal rule for approximating $\int_a^b f(x)\,dx$ using n trapezoids is given by

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)),$$

where $x_i = a + i \cdot \Delta x$ and $\Delta x = \frac{b-a}{n}$.

Example: Use the trapezoidal rule with $n = 3$ to approximate $\int_0^3 (x^2 + 1)\,dx$. Find the exact value of $\int_0^3 (x^2 + 1)\,dx$ and compare the results.

$f(x) = x^2 + 1$

$n = 3$

$a = 0$

$b = 3$

$\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$

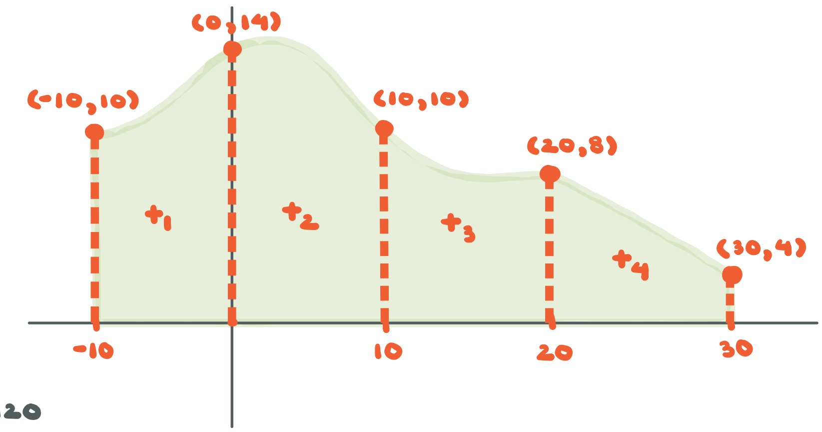
$x_i = a + i\Delta x = 0 + i \cdot 1 = i$

$$T_3 = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))$$
$$= \frac{1}{2} \cdot 1 (\underbrace{f(0)}_{1} + \underbrace{2f(1)}_{2 \cdot 2 = 4} + \underbrace{2f(2)}_{2 \cdot 5 = 10} + \underbrace{f(3)}_{10})$$
$$= \boxed{\frac{25}{2}}$$

$$\int_0^3 (x^2+1)\,dx = \frac{x^3}{3} + x \Big|_0^3 = \frac{3^3}{3} + 3 = \boxed{12}$$

nice estimate!

Example: Approximate the area of the shaded region using the trapezoidal rule with $n = 4$.



$f_1 = \frac{1}{2}\cdot 10(10+14) = 5\cdot 24 = 120$

$f_2 = \frac{1}{2}\cdot 10(14+10) = 120$

$f_3 = \frac{1}{2}\cdot 10(10+8) = 5\cdot 18 = 90$

$f_4 = \frac{1}{2}\cdot 10(8+4) = 5\cdot 12 = 60$

$f_1 + f_2 + f_3 + f_4 = \boxed{390}$

Example: Use the trapezoidal rule with $n = \cancel{4}\ 3$ to approximate $\int_{\cancel{0}\ \pi/4}^{\pi} \frac{\cos(x)}{x}\,dx$.

$f(x) = \frac{\cos(x)}{x}$

$n = 3$

$a = \pi/4$

$b = \pi$

$\Delta x = \frac{b-a}{n} = \frac{\pi - \pi/4}{3} = \frac{\pi}{4}$

$x_i = a + i\Delta x = \frac{\pi}{4} + i\frac{\pi}{4} = \frac{\pi}{4}(1+i)$

$T_3 = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$

$= \frac{1}{2}\cdot\frac{\pi}{4}\left[\underbrace{f\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}\cdot\frac{4}{\pi}} + \underbrace{2\cdot f\left(\frac{\pi}{2}\right)}_{2\cdot 0\cdot\frac{2}{\pi}} + \underbrace{2\cdot f\left(\frac{3\pi}{4}\right)}_{2\cdot\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{4}{3\pi}} + \underbrace{f(\pi)}_{-\frac{1}{\pi}}\right]$

$= \frac{\pi}{8}\Bigg[\underbrace{\frac{2\sqrt{2}}{\pi} + 0 - \frac{4\sqrt{2}}{3\pi} - \frac{1}{\pi}}_{\frac{6\sqrt{2} - 4\sqrt{2} - 3}{3\pi}}\Bigg]$

$= \boxed{\frac{2\sqrt{2} - 3}{24}}$

Example: Use the trapezoidal rule with $n = 3$ to approximate $\int_{-2}^{4} \sqrt{x^2+1}\, dx$.

$f(x) = \sqrt{x^2+1}$

$n = 3$

$a = -2$

$b = 4$

$\Delta x = \frac{b-a}{n} = \frac{4-(-2)}{3} = 2$

$x_i = a + i\Delta x = -2 + i2$

$T_3 = \frac{1}{2}\overset{2}{\Delta x}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$

$= f(-2) + 2f(0) + 2f(2) + f(4)$

$= \sqrt{5} + 2\cdot\sqrt{1} + 2\cdot\sqrt{5} + \sqrt{17}$

$= 3\sqrt{5} + \sqrt{17} + 2$