34 Lesson 34

34.1 Exponential Growth

Exponential Growth Model: Suppose y is a differentiable function of t such that $\frac{dy}{dt} = ky$, where k is a constant. Then,

$$y(t) = Ce^{kt},$$

where C is a constant. We say that k is the proportionality constant of growth rate. The constant C is the initial value of y.

Example: Given that $\frac{dy}{dt} = 5y$ and $y(2) = 10$, find $y(t)$. What is $y(4)$?

$\frac{dy}{dt} = 5y \Rightarrow y(t) = Ce^{5t}$

$10 = y(2) = Ce^{5\cdot 2} = Ce^{10} \Rightarrow C = \frac{10}{e^{10}}$

So, $y(t) = \frac{10}{e^{10}} e^{5t} = 10e^{5t-10}$

$y(4) = 10e^{5\cdot 4 - 10} = 10e^{10}$

Example: The rate of change of a population of a city is $\frac{dP}{dt} = kP$, where P is the population after t years.

$\frac{dP}{dt} = kP \rightarrow P(t) = Ce^{kt}$

1. If $P = 200$ when $t = 0$ and $P = 400$ when $t = 2$, what is $P(4)$?
2. If $P = 200$ when $t = 1$ and $P = 400$ when $t = 2$, what is $P(4)$?

(1) $P(0) = 200 \rightsquigarrow 200 = Ce^{k\cdot 0} = C \Rightarrow P(t) = 200e^{kt}$

$P(2) = 400 \rightsquigarrow 400 = 200e^{2k} \Rightarrow 2 = e^{2k}$

$\Rightarrow \ln(2) = \ln(e^{2k}) = 2k\ln(e) = 2k$ ($\ln(e) = 1$)

$\Rightarrow \frac{\ln(2)}{2} = k$

So, $P(t) = 200e^{\frac{\ln(2)}{2}t} = 200(e^{\ln(2)})^{t/2} = 200\cdot 2^{t/2}$

$P(4) = 200\cdot 2^{4/2} = 200\cdot 4 = 800$

② $P(1) = 200 \rightsquigarrow 200 = Ce^{k}$ (*)

$P(2) = 400 \rightsquigarrow 400 = Ce^{2k}$

Divide! $\frac{400}{200} = \frac{Ce^{2k}}{Ce^{k}} \Rightarrow 2 = e^{2k-k} = e^{k}$

$\Rightarrow \ln(2) = \ln(e^{k}) = k\underbrace{\ln(e)}_{1} = k$

So, plugging into (*), $200 = Ce^{\ln(2)} = 2C \Rightarrow C = 100$

Hence, $P(t) = 100e^{\ln(2)t}$

$P(4) = 100(e^{\ln(2)})^{4} = 100 \cdot 2^{4} = 1600$

Example: Suppose I invest \$500 into a savings account where the interest is compounded continuously. After 3 years, there is \$700 in my account.

1. What is the annual interest rate? ← Find k.
2. How much money is in the account after 6 years?

① $C = 500 \rightsquigarrow y(t) = 500e^{kt}$

$y(3) = 700 \Rightarrow 700 = y(3) = 500e^{3k}$

$\Rightarrow \frac{700}{500} = e^{3k}$

$\Rightarrow \ln(7/5) = \ln(e^{3k}) = 3k\underbrace{\ln(e)}_{1} = 3k$

$\Rightarrow \frac{\ln(7/5)}{3} = k$

$\approx .112$

② $y(t) = 500e^{\left[\frac{\ln(7/5)}{3}\right]t}$

$y(6) = 500 \cdot \left[e^{\ln(7/5)}\right]^{2}$

$= 500(7/5)^{2}$

$= 500 \cdot \frac{49}{25}$

$= 20 \cdot 49$

$= \$980$