35 Lesson 35

35.1 Exponential Decay

If y is a differentiable function of t such that $y'(t) = ky$ for some constant k, then

$$y = Ce^{kt},$$

where C is a constant. The constant k is the proportionality constant or decay rate. The constant C is the initial value of y. If k is negative, this model is the exponential decay (or negative growth) model.

Examples of Applications: Decay of population and radioactive isotopes

Example: Suppose $\frac{dy}{dt} = -2y$ and $y(15) = 100$. Find $y(20)$.

$$y(t) = Ce^{-2t}$$

$$100 = y(15) = Ce^{-2\cdot 15} = Ce^{-30}$$

$$\Rightarrow \frac{100}{e^{-30}} = C$$

$$\Rightarrow 100e^{30} = C$$

$$y(t) = 100e^{30}e^{-2t} = 100e^{30-2t}$$

$$y(20) = 100e^{-10} = \boxed{\frac{100}{e^{10}}}$$

Example: The population of a species of fish is decreasing at a rate that is proportional to the population itself. If $P = 800$ when $t = 2$ and $P = 600$ when $t = 4$, what is the population when $t = 8$?

$$\frac{dP}{dt} = kP \Rightarrow P(t) = Ce^{kt}$$

$$P(2) = 800 \rightsquigarrow 800 = P(2) = Ce^{2k} \quad (*)$$

$$P(4) = 600 \rightsquigarrow 600 = P(4) = Ce^{4k}$$

Divide! $$\frac{600}{800} = \frac{\cancel{C}e^{4k}}{\cancel{C}e^{2k}}$$

$$\Rightarrow \frac{3}{4} = e^{2k}$$

$$\Rightarrow \ln(3/4) = \ln(e^{2k})$$

$$\Rightarrow \ln(3/4) = 2k\overbrace{\ln(e)}^{1}$$

$$\Rightarrow \frac{1}{2}\ln(3/4) = k$$

Plug into $(*)$.

$$800 = Ce^{2\cdot\frac{1}{2}\ln(3/4)}$$

$$\Rightarrow 800 = C\cancel{e^{\ln}}^{(3/4)}$$

$$\Rightarrow 800 = C\cdot\frac{3}{4}$$

$$\Rightarrow 800\cdot\frac{4}{3} = C$$

$$\Rightarrow \frac{3200}{3} = C$$

So, $$P(t) = \frac{3200}{3}\cdot e^{\frac{1}{2}\ln(3/4)t}$$

$$P(8) = \frac{3200}{3}e^{4\ln(3/4)} = \frac{3200}{3}\left(\cancel{e^{\ln}}^{(3/4)}\right)^4 = \frac{3200}{3}\cdot\frac{3^4}{4^4}$$

$$= \frac{3200\cdot 3^3}{4^4} = \frac{200\cdot 3^3}{16} = \boxed{337.5 \text{ fish}}$$

Radioactive Isotopes and Half-life

Radioactive isotopes decay over time, and they follow the exponential decay model. The decay rate is distinct for each isotope, and it is characterized by half-life. The half-life of an isotope is the time it takes for the isotope to reduce to half of its original quantity. The decay rate k of an isotope can be found using its half-life, namely

$$k = \frac{\ln\left(\frac{1}{2}\right)}{\text{half-life}}.$$

Example: The radioactive isotope ^{14}C (carbon-14) has a half-life of approximately 5,715 years. Suppose that after 2,500 years, there are 2 grams of carbon-14 left.

1. How much carbon-14 (in grams) was there initially? $\rightarrow$ Find C.
2. How much carbon-14 is there after 10,000 years?

$$k = \frac{\ln(1/2)}{5715}$$

$$y(t) = Ce^{kt}$$

1
$$2 = y(2500) = Ce^{\left[\frac{\ln(1/2)}{5715}\right]2500}$$

$$\Rightarrow \frac{2}{e^{\left[\frac{\ln(1/2)}{5715}\right]2500}} = C$$

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$\boxed{2.7084 \text{ grams}}$

2
$$y(t) = \frac{2}{e^{\left[\frac{\ln(1/2)}{5715}\right]2500}} e^{\frac{\ln(1/2)}{5715}t}$$

$\boxed{y(10000) \approx .8053 \text{ grams}}$

Example: The radioactive isotope ^{226}Ra (radium-226) has a half-life of 1,599 years. A piece of ancient material contains only 40% as much of the radioactive radium-226 as a piece of modern material. How old is the ancient sample of radium-226?

40% of initial amount
= 40% of C
= .40 · C

$$K = \frac{\ln(1/2)}{1599}$$

$$.4C = Ce^{\frac{\ln(1/2)}{1599}t}$$

$$\Rightarrow \quad .4 = e^{\frac{\ln(1/2)}{1599}t}$$

$$\Rightarrow \quad \ln(.4) = \ln\left(e^{\frac{\ln(1/2)}{1599}t}\right)$$

$$= \frac{\ln(1/2)}{1599}t \underbrace{\ln(e)}_{1}$$

$$\Rightarrow \quad \ln(.4) \cdot \frac{1599}{\ln(1/2)} = t$$

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$$\boxed{2113.76 \text{ years old}}$$