3 Lesson 3

3.1 Finding Limits Analytically

To find $\lim_{x\to c} f(x)$ analytically, we consider $f(c)$ (i.e. we plug $x = c$ into $f(x)$). We will examine three possible scenarios.

<u>**Case 1:**</u> If $f(c)$ returns a number (0 is okay), then $\lim_{x\to c} f(x) = f(c)$.

Example 1: Evaluate the following limit analytically:

$$\lim_{x\to 3}(2x^2 + 7) = 25$$

$c = 3$

$f(x) = 2x^2 + 7$

$f(c) = f(3) = 2\cdot 3^2 + 7 = 18 + 7 = 25$

<u>**Case 2:**</u> If $f(c) = \frac{\text{nonzero number}}{0}$, $f(x)$ has a vertical asymptote at $x = c$. So, $\lim_{x\to c} f(x)$ can equal ∞, $-\infty$, or DNE. We need to evaluate the left-sided and right-sided limits to determine $\lim_{x\to c} f(x)$.

Example 2: Evaluate the following limit analytically:

$$\lim_{x\to -1} \frac{10}{(x+1)^2} = \infty$$

$c = -1$

$f(x) = \frac{10}{(x+1)^2}$

$f(c) = f(-1) = \frac{10}{(-1+1)^2} = \frac{10}{0}$

$\lim_{x\to -1^-} \frac{10}{(x+1)^2} = \infty$ $\qquad \frac{10}{(\text{small negative \#})^2} = \text{really big positive \#}$

$\lim_{x\to -1^+} \frac{10}{(x+1)^2} = \infty$ $\qquad \frac{10}{(\text{small positive \#})^2} = \text{really big positive \#}$

Example 3: Evaluate the following limit analytically:

$$\lim_{x\to 3} \frac{1}{x-3} = \text{DNE}$$

$c = 3$

$f(x) = \frac{1}{x-3}$

$f(3) = \frac{1}{3-3} = \frac{1}{0}$

$\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$ $\qquad \frac{1}{\text{small negative \#}} = \text{really big negative \#}$

$\lim_{x\to 3^+} \frac{1}{x-3} = \infty$ $\qquad \frac{1}{\text{small positive \#}} = \text{really big positive \#}$

Case 3: If $f(c) = \frac{0}{0}$, $f(x)$ has a hole or vertical asymptote at $x = c$. We need to manipulate $f(x)$ (using algebra) so that $f(c)$ returns a number or $\frac{\text{nonzero number}}{0}$ (i.e. case 1 or 2).

Example 4: Evaluate the following limit analytically:

$$\lim_{x \to 0} \frac{x^3 - 6x^2}{x^3 + 3x^2} = -2$$

$c = 0$

$f(x) = \frac{x^3 - 6x^2}{x^3 + 3x^2}$

$f(0) = \frac{0}{0}$

$f(x) = \frac{x^3 - 6x^2}{x^3 + 3x^2} = \frac{x^2(x-6)}{x^2(x+3)} = \frac{x-6}{x+3}$

$\lim_{x \to 0} \frac{x^3 - 6x^2}{x^3 + 3x^2} = \lim_{x \to 0} \frac{x-6}{x+3} \rightsquigarrow \frac{0-6}{0+3} = \frac{-6}{3} = -2$ (case 1)

Example 5: Evaluate the following limit analytically:

$$\lim_{x \to 1} \frac{x^2 - x}{(x-1)^2} = \text{DNE}$$

$c = 1$

$f(x) = \frac{x^2 - x}{(x-1)^2}$

$f(1) = \frac{0}{0}$

$f(x) = \frac{x^2 - x}{(x-1)^2} = \frac{x(x-1)}{(x-1)^2} = \frac{x}{x-1}$

$\lim_{x \to 1} \frac{x^2 - x}{(x-1)^2} = \lim_{x \to 1} \frac{x}{x-1}$

$\frac{1}{1-1} = \frac{1}{0}$ (case 2)

$\lim_{x \to 1^-} \frac{x}{x-1} = -\infty$

$\lim_{x \to 1^+} \frac{x}{x-1} = \infty$

We can also find limits of piecewise functions analytically.

Example 6: Suppose

$$f(x) = \begin{cases} x + 3 & \text{if } x \leq 0, \\ x^2 + 3 & \text{if } 0 < x < 1, \\ 2x + 1 & \text{if } x \geq 1. \end{cases}$$

Evaluate the following limits analytically:

$\lim_{x \to 0^-} f(x) = 3$

$\lim_{x \to 0^+} f(x) = 3$

$\lim_{x \to 0} f(x) = 3$

$\lim_{x \to 1^-} f(x) = 4$

$\lim_{x \to 1^+} f(x) = 3$

$\lim_{x \to 1} f(x) = \text{DNE}$

3.2 Limit Properties

Let c, k, L, and M be real numbers and n a positive integer. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then the following hold:

- $\lim_{x\to c}[kf(x)] = kL$
- $\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$
- $\lim_{x\to c}[f(x)g(x)] = LM$
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, if $M \neq 0$
- $\lim_{x\to c}[f(x)]^n = L^n$

Example 7: Suppose $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = 4$. Evaluate

$$\lim_{x\to 1}[5f(x) - g(x)^2 + 7]$$

using the limit properties above.

$$\lim_{x\to 1}[5f(x) - g(x)^2 + 7] = \lim_{x\to 1} 5f(x) - \lim_{x\to 1} g(x)^2 + \lim_{x\to 1} 7$$

$$= 5\lim_{x\to 1} f(x) - (\lim_{x\to 1} g(x))^2 + 7$$

$$= 5 \cdot 2 - 4^2 + 7$$

$$= 10 - 16 + 7$$

$$= 1$$