# 4 Lesson 4

## 4.1 Continuity

**Idea:** a function is continuous if we can draw its graph without lifting our pen.

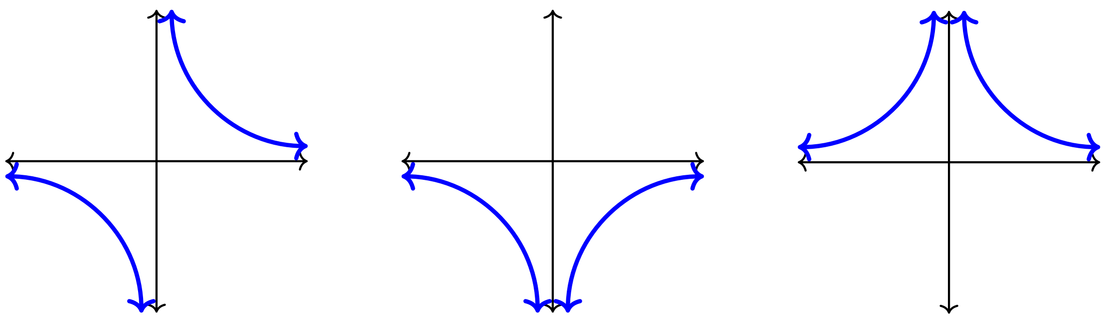
**Definition:** A function $f(x)$ is continuous at $x = c$ if the following hold:

- $f(c)$ is defined
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

If any of the above conditions fail, we say $f(x)$ is discontinuous at $x = c$.

## 4.2 Types of Discontinuities

**Vertical Asymptote:** A function $f(x)$ has a vertical asymptote at $x = c$ if $\lim_{x \to c^-} f(x) = \pm\infty$ and/or $\lim_{x \to c^+} f(x) = \pm\infty$.

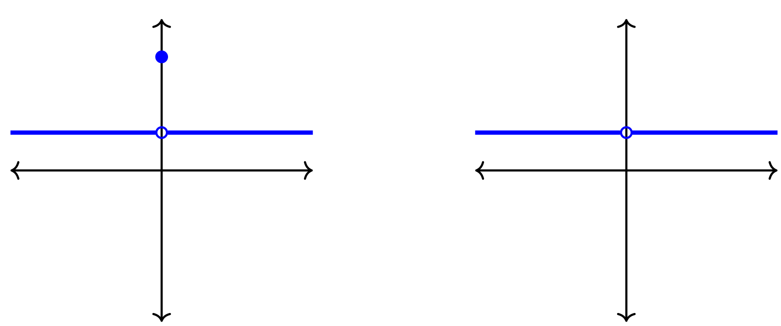


Using terminology from lesson 3, $f(x)$ has a vertical asymptote at $x = c$ when $\lim_{x \to c} f(x)$ is either a "Case 2" limit *or* a "Case 3" limit that becomes a "Case 2" limit when $f(x)$ is simplified using algebraic manipulation.

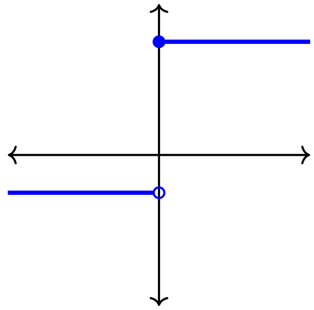
**Hole:** A function $f(x)$ has a hole at $x = c$ if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x) = L < \infty \quad \text{and} \quad f(c) \neq L.$$

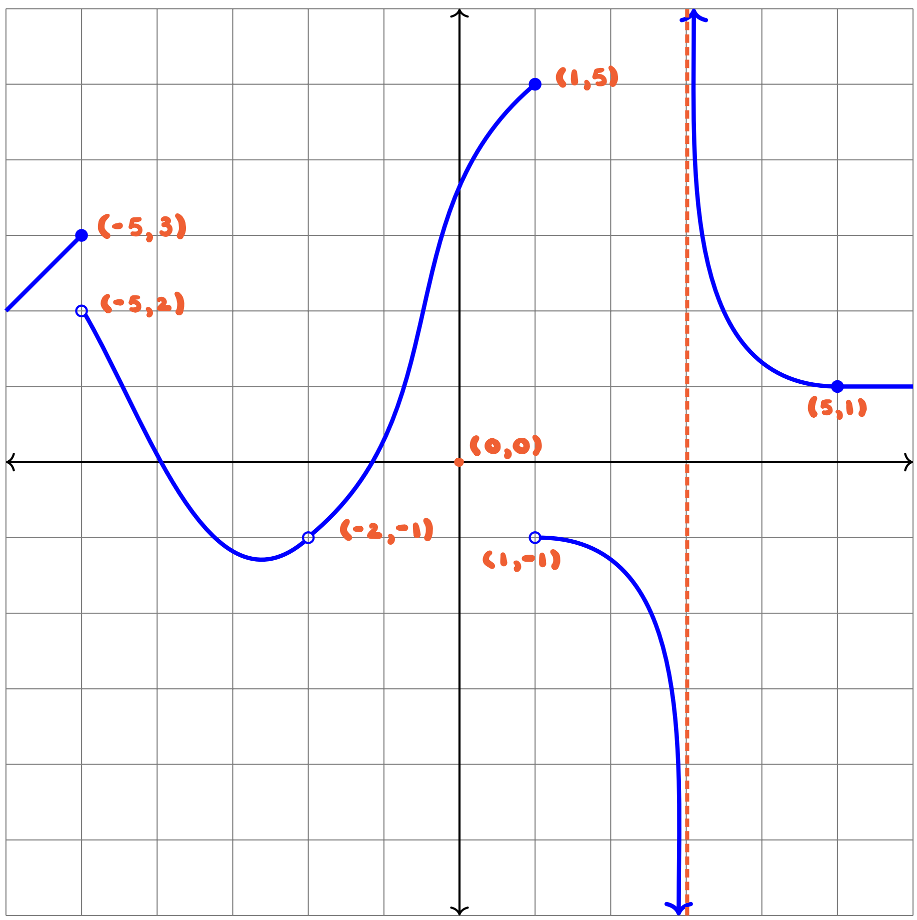
Note that $f(c)$ can be undefined. Using terminology from lesson 3, holes can show up as "Case 3" limits that become "Case 1" limits.



**Jump:** A function $f(x)$ has a jump at $x = c$ if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = M$ where $L, M < \infty$ and $L \neq M$.



**Example 1:** Given the graph of $f(x)$ below, find the $x$ values where $f(x)$ is discontinuous. Classify the discontinuities.




- jump at $x=-5$
- hole at $x=-2$
- jump at $x=1$
- $\underbrace{\text{vertical asymptote}}_{\text{V.A.}}$ at $x=3$

**Example 2:** Classify the discontinuities, if any, of the following function:

$$f(x) = \frac{x^2 - 5x}{x^2 + 3x}$$

rational function $\Rightarrow$ see which x-values make the denomenator 0

$x^2 + 3x = 0 \Rightarrow x(x+3) = 0 \Rightarrow$ check at $x=0$ and $x=-3$

Let's determine the type of limit $f(x)$ has at $x=0$ and $x=-3$ to determine the type of discontinuity at each point.

$\underline{x=-3}$

$$\lim_{x \to -3} \frac{x^2-5x}{x^2+3x}$$

plug in $x=-3 \rightsquigarrow \frac{(-3)^2 - 5(-3)}{(-3)^2 + 3(-3)} = \frac{9+15}{9-9} = \frac{24}{0}$ (case 2!) $\Rightarrow$ V.A. at $x=-3$

$\underline{x=0}$

$$\lim_{x \to 0} \frac{x^2-5x}{x^2+3x}$$

plug in $x=0 \rightsquigarrow \frac{0}{0}$ (case 3!) $\Rightarrow$ simplify $\frac{x^2-5x}{x^2+3x} = \frac{x(x-5)}{x(x+3)} = \frac{x-5}{x+3}$

$$\lim_{x \to 0} \frac{x^2-5x}{x^2+3x} = \underbrace{\lim_{x \to 0} \frac{x-5}{x+3}}_{\substack{\text{plug in } x=0 \\ \text{into } \frac{x-5}{x+3} \rightsquigarrow -\frac{5}{3} \text{ (case 1!)}}} = -\frac{5}{3} \Rightarrow \text{hole at } x=0$$

**Example 3:** Classify the discontinuities, if any, of the following function:

$$g(x) = x^4 + 11x^2 - 8x + 22$$

polynomials are continuous everywhere $\Rightarrow$ no discontinuities

**Example 4:** Classify the discontinuities, if any, of the following function:

$$h(x) = \begin{cases} 7x+1 & \text{if } x \neq 1, \\ 6 & \text{if } x = 1. \end{cases}$$

7x+1 and 6 are both continuous, so we only need to check x-values where h(x) changes definition, i.e. x=1.

$$\left.\begin{aligned} &\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} (7x+1) = 8 \\ &\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (7x+1) = 8 \\ &h(1) = 6 \end{aligned}\right\}$$

hole at x=1

**Example 5:** Classify the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} 8x^2+2 & \text{if } x \leq 0, \\ 3x+2 & \text{if } 0 < x < 1, \\ x+9 & \text{if } x \geq 1. \end{cases}$$

<u>x=0</u>

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (8x^2+2) = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x+2) = 2$

$\lim_{x \to 0} f(x) = 2$

$f(0) = 8 \cdot 0^2 + 2 = 2$

$\Rightarrow$ f(x) continuous at x=0

<u>x=1</u>

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x+2) = 5$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x+9) = 10$

$\Rightarrow$ jump at x=1