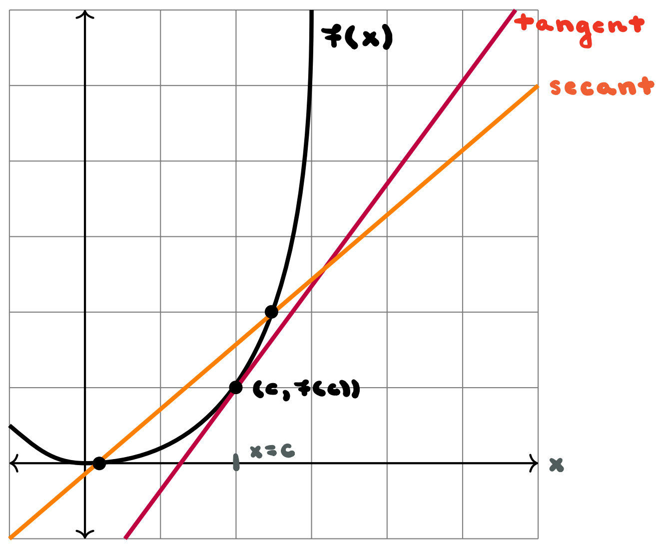
5 Lesson 5

5.1 Secant and Tangent Lines

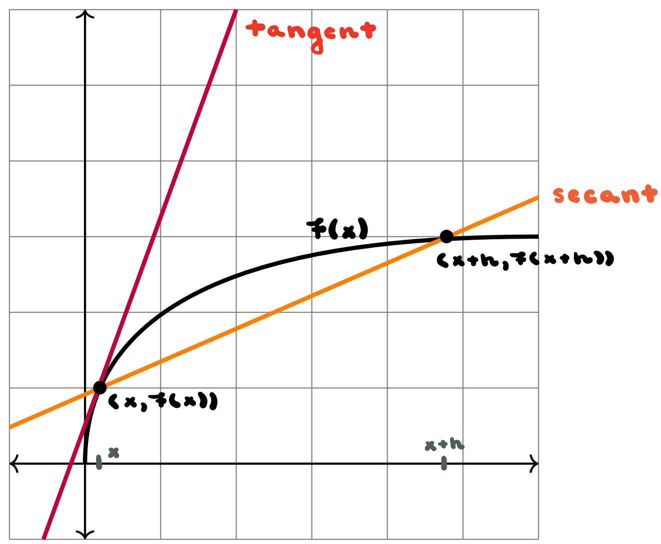
Definition: A secant line of a function $f(x)$ is a line that passes through two distinct points of $f(x)$.

Definition: The tangent line to a function $f(x)$ at the point $x = c$ is the line that touches the graph of $f(x)$ at the point $(c, f(c))$. We can think of this as taking the slope of the graph at the point $(c, f(c))$ and extending it into a line with the same slope.



Note: The slope of the tangent line to $f(x)$ at $x = c$ is the slope of the graph of $f(x)$ at the point $(c, f(c))$.

5.2 Finding the Slope of the Tangent Line Using Slopes of Secant Lines



What is the slope of the secant line in the above picture?

Recall: slope of line formed by points (x_1, y_1) and (x_2, y_2) $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\text{slope of secant line} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$ $\leftarrow$ call this the difference quotient

Take the limit as $h \to 0$ to get the slope of the tangent line in the above picture.

$$\text{slope of tangent line} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5.3 The Derivative

Definition: The derivative of a function $f(x)$ at the point x is the slope of the tangent line to $f(x)$ at the point x. We write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Notation: If $y = f(x)$, then we can also write $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x)$.

Geometric Interpretation: The slope of the tangent line to $f(x)$ at the point $x = c$ is equal to the derivative of $f(x)$ at the point $x = c$, i.e. $f'(c)$ is the slope of the tangent line to $f(x)$ at $x = c$.

Example 1: Suppose $f(x) = x + 5$. Find $f'(x)$.

$f(x+h) = (x+h)+5$

$f(x+h) - f(x) = (x+h)+5-(x+5) = h$

$$\frac{f(x+h)-f(x)}{h} = \frac{h}{h} = 1$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 1 = 1$$

$\Rightarrow$ $f'(x) = 1$

Example 2: Suppose $f(x) = 3x^2 + 1$. Find $f'(x)$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2+1-(3x^2+1)}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2hx+h^2)+1-3x^2-1}{h}$$

$$= \lim_{h\to 0} \frac{6hx+3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(6x+3h)}{h}$$

$$= \lim_{h\to 0} 6x+3h$$

$$= 6x$$

$\Rightarrow$ $f'(x) = 6x$

Example 3: Find the equation of the tangent line to the graph of $f(x) = 3x^2 + 1$ at the point $x = 2$.

to find the equation of a line, we need a point on the line and the slope

From example 2, $f'(x) = 6x$. So, the slope of the tangent line to $f(x)$ at $x=2$ is $f'(2)$. Note $f'(2) = 6 \cdot 2 = 12$.

slope

We know the x-coordinate of the point is 2, so the y-coordinate is $f(2)$. Note $f(2) = 3 \cdot 2^2 + 1 = 13$. $(2,13)$ is a point on the line.

Recall the point-slope form of a line with slope m and point (x_1, y_1) is: $y - y_1 = m(x - x_1)$.

So, we have $y - 13 = 12(x-2)$

$\Rightarrow$ $y = 12x - 24 + 13$

$\Rightarrow$ $y = 12x - 11$

Example 4: The derivative of a function $f(x)$ is found by computing

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^3 + 2} - \sqrt{x^3+2}}{h}.$$

What is $f(x)$?

compare

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sqrt{x^3+2}$$