6 Lesson 6

6.1 Basic Rules of Differentiation

(1) Constant Rule: For any constant c,

$$\frac{d}{dx}(c) = 0.$$

(2) Power Rule: For any real number n,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(3) Constant Multiple Rule: Let c be a constant and $f(x)$ differentiable. Then,

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)).$$

(4) Sum/Difference Rule: Suppose $f(x)$ and $g(x)$ are differentiable. Then,

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)).$$

Proof of (1): Let's use the definition of a derivative from lesson 5. Let $f(x) = c$. Then,

$$\frac{d}{dx}(c) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Note: Recall that $x^{-n} = \frac{1}{x^n}$ and $x^{n/m} = \sqrt[m]{x^n}$.

Example 1: Find the derivatives of the following functions.

(a) $f(x) = 75$

$$f'(x) = \frac{d}{dx}(75) = 0$$

constant rule

(b) $f(x) = x^3$

$$f'(x) = \frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

power rule

(c) $f(x) = \frac{1}{x^4} = x^{-4}$

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}$$

power rule

(d) $f(x) = \sqrt[3]{x^2} = x^{2/3}$

$$f'(x) = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

power rule

(e) $f(x) = 4x^3$

power rule

$$f'(x) = \frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3) = 4(3x^{3-1}) = 4\cdot 3x^2 = 12x^2$$

constant multiple rule

(f) $f(x) = x^5 + 2x - 3$

$$f'(x) = \frac{d}{dx}(x^5+2x-3) = \frac{d}{dx}(x^5) + \frac{d}{dx}(2x) - \frac{d}{dx}(3)$$ ← sum/diff. rule

$$= 5x^4 + 2 + 0$$ ← power rule twice and constant rule

$$= 5x^4 + 2$$

6.2 Sine and Cosine

We note that

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x).$$

Example 2: Suppose $f(x) = 2\sin(x) - 3\cos(x)$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}(2\sin(x) - 3\cos(x)) = \frac{d}{dx}(2\sin(x)) - \frac{d}{dx}(3\cos(x))$$ ← sum/diff. rule

$$= 2\frac{d}{dx}(\sin(x)) - 3\frac{d}{dx}(\cos(x))$$ ← constant multiple rule twice

$$= 2\cos(x) - 3\cdot(-\sin(x))$$

$$= 2\cos(x) + 3\sin(x)$$

Example 3: Find the equation of the tangent line to $f(x) = 2\sin(x) - 3\cos(x)$ at $x = \frac{\pi}{2}$.

By example 2, $f'(x) = 2\cos(x) + 3\sin(x)$.

slope of tangent line: $f'(\pi/2) = 2\cos(\pi/2) + 3\sin(\pi/2) = 2\cdot 0 + 3\cdot 1 = 3$

point on tangent line: $(\pi/2, f(\pi/2)) = (\pi/2, 2)$

↳ $f(\pi/2) = 2\sin(\pi/2) - 3\cos(\pi/2) = 2\cdot 1 - 3\cdot 0 = 2$

So, $y - 2 = 3(x - \pi/2)$

$\Rightarrow y = 3x - 3\pi/2 + 2$

$\Rightarrow y = 3x + \frac{4-3\pi}{2}$

6.3 Exponential Function

We note that

$$\frac{d}{dx}(e^x) = e^x.$$

Example 4: Suppose $f(x) = 7e^x$. Find $f'(x)$.

$f'(x) = \frac{d}{dx}(7e^x) = 7\frac{d}{dx}(e^x) = \boxed{7e^x}$

constant multiple rule

6.4 Practice Problems

(1) Suppose $f(x) = (x-1)(x+3)$. Find all the values of x so that $f'(x) = 1$.

$f(x) = x^2 + 3x - x - 3 = x^2 + 2x - 3$

$f'(x) = 2x + 2$

Want to find x-values s.t. $1 = f'(x) = 2x + 2$

Solving for x: $1 = 2x + 2$

$\Rightarrow -1 = 2x$

$\Rightarrow \boxed{x = -1/2}$

(2) Suppose $f(x) = \frac{x^2 - 2x^{3/2}}{\sqrt{x}}$. Find $f'(x)$.

$$f(x) = \frac{x^2 - 2x^{3/2}}{\sqrt{x}}$$

$$= \frac{x^2}{\sqrt{x}} - \frac{2x^{3/2}}{\sqrt{x}}$$

$$= \frac{x^2}{x^{1/2}} - \frac{2x^{3/2}}{x^{1/2}}$$

$$= x^{3/2} - 2x$$

$$f'(x) = \frac{3}{2}x^{1/2} - 2$$

$$= \boxed{\frac{3}{2}\sqrt{x} - 2}$$