8 Lesson 8

8.1 The Product Rule

Example 1: Suppose $f(x) = 3x^3(5x^2 + 7x)$. Find $f'(x)$.

$f(x) = 15x^5 + 21x^4$

$f'(x) = 75x^4 + 84x^3$

What happens if we take the derivative of each factor and multiply them together?

$f'(x) \overset{?}{=} 9x^2(10x + 7) = 90x^3 + 63x^2$ ← NO!

$$\frac{d}{dx}[f(x)g(x)] \neq \left[\frac{d}{dx}f(x)\right]\left[\frac{d}{dx}g(x)\right]$$

Using the limit definition of a derivative (and some cleverness), we can show that

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + \left[\frac{d}{dx}g(x)\right]f(x). \quad (*)$$

Example 1 (revisited): Suppose $y = 3x^3(5x^2 + 7x)$. Find y' using the product rule.

$f(x) = 3x^3$

$g(x) = 5x^2 + 7x$

$f'(x) = 9x^2$

$g'(x) = 10x + 7$

$y' = f'(x)g(x) + g'(x)f(x)$

$= 9x^2(5x^2 + 7x) + (10x + 7)3x^3$

$= 45x^4 + 63x^3 + 30x^4 + 21x^3$

$= 75x^4 + 84x^3$

Example 2: Find the derivative of $y = 3e^x \cos(x)$ at $x = \pi$.

$f(x) = 3e^x$

$g(x) = \cos(x)$

$f'(x) = 3e^x$

$g'(x) = -\sin(x)$

$y' = f'(x)g(x) + g'(x)f(x)$

$= 3e^x\cos(x) - \sin(x)\cdot 3e^x$

$= 3e^x(\cos(x) - \sin(x))$

$y'(\pi) = 3e^\pi(\underbrace{\cos(\pi)}_{-1} - \underbrace{\sin(\pi)}_{0}) = -3e^\pi$

8.2 Practice Problems

1. Find the x-values at which $y = 6x^8e^x$ has a horizontal tangent line.

where tangent line has slope 0, i.e. where deriv. is 0

$f(x) = 6x^8$

$g(x) = e^x$

$y' = 48x^7e^x + e^x 6x^8$

$= 6x^7e^x(8+x)$

Solve $0 = 6x^7e^x(8+x)$

$\Rightarrow 6x^7e^x = 0$

$8 + x = 0$

$\Rightarrow x = 0$, $x = -8$

2. Find the equation of the tangent line to the curve of $y = 7x\sin(x)$ at $x = \pi$.

$y' = 7\sin(x) + \cos(x)7x$

$y'(\pi) = 7\underbrace{\sin(\pi)}_{0} + \underbrace{\cos(\pi)}_{-1}\cdot 7\pi = -7\pi \leftarrow$ slope

$y(\pi) = 7\pi\sin(\pi) = 0$

$\hookrightarrow$ pt. is $(\pi, 0)$

$y - 0 = -7\pi(x - \pi)$

$\Rightarrow y = -7\pi x + 7\pi^2$

proof of (*):

def. of derivative

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

add "0"

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x+h)g(x) + f(x+h)g(x)}^{=0} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0} g(x)\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] + g(x) \cdot \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= f(x)g'(x) + g(x)f'(x)$$